

Proof by Induction

Introduction

Let us consider the following two expressions:

$$1^3 + 2^3 + 3^3 + \dots + n^3 \quad \text{and} \quad \frac{1}{4}n^2(n+1)^2.$$

Or in slightly more elegant notation

$$\sum_{k=1}^n k^3 \quad \text{and} \quad \frac{1}{4}n^2(n+1)^2.$$

The following table shows the results for the first few values of n

n	$\sum_{k=1}^n k^3$	$\frac{1}{4}n^2(n+1)^2$
1	1	1
2	9	9
3	36	36
4	100	100
5	225	225
6	441	441

So we can see that it *appears* that the two terms are equal. However a Mathematician should not be happy with this. How do we know that when n is 100,000 or 100,000,000 the two terms will still be equal? The answer is that we don't. But with a process known as *Proof by Induction* we can prove that the two terms will always be the same for all n .

Theory

Proof by Induction is a method of proving that two terms will always be equal for any value of n . The basic method is as follows

1. Demonstrate that the two terms are the same for some starting value of n . Usually one uses $n = 1$ or $n = 0$, but you can start at any n you fancy.
2. Assume the equality holds for $n = k$.
3. Try and show that *if* it is true for $n = k$ *then* it is also true for $n = k + 1$.

If it is true for some fixed value (STEP 1) and *if* it is true for $n = k$ *then* it is true for $n = k + 1$ then we will have proved that it is true for all values of n greater than or equal to the one chosen in STEP 1. Think of the following diagrammatic representation:

$$\boxed{n = k \Rightarrow n = k + 1} \text{ so } \boxed{n = 1 \Rightarrow n = 2 \Rightarrow n = 3 \Rightarrow n = 4 \Rightarrow n = 5 \Rightarrow n = 6 \Rightarrow n = 7 \dots}$$

Let us try and use this technique to try and prove $1 + 2 + 3 + \dots + n = \frac{1}{2}n(n+1)$; a very well known result from C2.

1. The statement is true for $n = 1$ because both sides of the statement are clearly 1.
2. Assume the equality holds for k . i.e. $1 + 2 + 3 + \dots + k = \frac{1}{2}k(k+1)$.

3. We therefore *wish to show* that $1 + 2 + 3 + \dots + k + (k + 1) = \frac{1}{2}(k + 1)(k + 2)$ (the formula we would expect for $k + 1$). We therefore take

$$\begin{aligned} 1 + 2 + 3 + \dots + k + \underline{(k + 1)} &= \frac{1}{2}k(k + 1) + \underline{(k + 1)} \\ &= \frac{1}{2}[k(k + 1) + 2(k + 1)] \\ &= \frac{1}{2}(k + 1)(k + 2). \end{aligned}$$

Which is what we wanted, so since the formula is true for $n = 1$ and *if* it is true for $n = k$ *then* it is also true for $n = k + 1$ we can conclude that the formula is true for all $n \geq 1$. And we are done.

Example

1. The example at the beginning; prove that $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{1}{4}n^2(n + 1)^2$.

Well firstly the equation clearly holds for $n = 1$ (both sides are 1).

We assume it holds for $n = k$, so $1^3 + 2^3 + 3^3 + \dots + k^3 = \frac{1}{4}k^2(k + 1)^2$.

We therefore need to show that $1^3 + 2^3 + 3^3 + \dots + k^3 + (k + 1)^3 = \frac{1}{4}(k + 1)^2(k + 2)^2$.

So the algebraic slog begins:

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \dots + k^3 + \underline{(k + 1)^3} &= \frac{1}{4}k^2(k + 1)^2 + \underline{(k + 1)^3} \\ &= \frac{1}{4}[k^2(k + 1)^2 + 4(k + 1)^3] \\ &= \frac{1}{4}[(k + 1)^2(k^2 + 4(k + 1))] \\ &= \frac{1}{4}[(k + 1)^2(k^2 + 4k + 4)] \\ &= \frac{1}{4}(k + 1)^2(k + 2)^2. \end{aligned}$$

Which is what we wanted, so we are done. True for $n = 1$. If true for $n = k$ then true for $n = k + 1$, so true for all $n \geq 1$.