C4 Numerical Integration

Patrons are reminded that definite integrals (i.e. areas under graphs) can be approximated by the trapezium rule,

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{h}{2}\left[y_0 + y_n + 2(y_1 + \cdots + y_{n-1})\right],$$

where h is the 'width' of each trapezium and the y values are the 'heights' of the parallel sides.

On the whole, the approximation can be improved by increasing the number of trapezia (although this is not always the case).

Patrons are also reminded that percentage error is defined:

$$\text{Percentage Error} \equiv \frac{|\text{Approximate Value} - \text{True Value}|}{\text{True Value}} \times 100.$$

1. (a) Find the exact value of $\int_0^\pi \sin x\,dx$. $\boxed{2}$
   (b) Using four trapezia find an approximate value of $\int_0^\pi \sin x\,dx$. $\boxed{1.896}$
   (c) Explain why your approximate value is smaller than the true value.

2. (a) Find the exact value of $\int_1^4 3e^{2x}\,dx$. $\boxed{\frac{3e^2(e^6-1)}{2}}$
   (b) Using three trapezia find an approximate value of $\int_1^4 3e^{2x}\,dx$. $\boxed{5857}$
   (c) Find the percentage error in your approximation. $\square$

3. (a) Split $\dfrac{3x+5}{x^2+4x+3}$ into partial fractions. $\boxed{\frac{1}{x+1} + \frac{2}{x+3}}$
   (b) Hence, find the *exact* value of $\displaystyle\int_0^4 \frac{3x+5}{x^2+4x+3}\,dx$, giving your answer as a single logarithm . $\boxed{\ln\left(\frac{245}{9}\right)}$
   (c) Using four trapezia find an approximate value of $\displaystyle\int_0^4 \frac{3x+5}{x^2+4x+3}\,dx$. Give your approximation to three significant figures. $\boxed{3.39}$

4. (a) Find the *exact* value of $\displaystyle\int_2^5 \frac{x}{\sqrt{x^2+1}}\,dx$. $\boxed{\sqrt{26}-\sqrt{5}}$
   (b) Using three trapezia find an approximate value of $\displaystyle\int_2^5 \frac{x}{\sqrt{x^2+1}}\,dx$. Give your approximation to three significant figures. $\boxed{2.86}$
   (c) By considering a sketch of $y = \dfrac{x}{\sqrt{x^2+1}}$, explain why your approximation is an underestimate of the true value.

5. (a) Split $\dfrac{3x^2+9x+2}{x^2+2x}$ into partial fractions. $\boxed{3 + \frac{1}{x} + \frac{2}{x+2}}$

(b) Hence, find the *exact* value of $\int_5^{10} \frac{3x^2+9x+2}{x^2+2x}\,dx$, giving your answer in the form $a+\ln\left(\frac{b}{c}\right)$, where a, b and c are integers. $15+\ln(\frac{288}{49})$

(c) Using five trapezia find an approximate value of $\int_5^{10} \frac{3x^2+9x+2}{x^2+2x}\,dx$. Give your approximation to seven decimal places (I know that's ridiculous, but do it anyway). 16.7758658

(d) Find the percentage error in your approximation, to two significant figures. 0.028% error