Simultaneous Equations

Consider the following equations;

$$\begin{aligned} 3x + 2y &= 7 \\ 2x - y &= 0 \end{aligned}.$$

We know that both represent a straight line in the xy-plane. If the lines have different gradient then they will cross at *one* point. This point (x, y) represents where *both* equations are solved *simultaneously* $\Rightarrow$ SIMULTANEOUS EQUATIONS.

We *could* draw the straight line graphs for both lines and find where they cross, but this is time consuming and inaccurate. Simultaneous equations can be solved two ways; by substitution and elimination. I am going to explain the elimination method here.

We must try to get the same number of x's or y's in both equations. We do this by multiplying one or both of the equation(s) by a cunningly chosen number. So in the above example we could either make the x's the same as follows:

$$\begin{matrix} 3x + 2y = 7 \\ 2x - y = 0 \end{matrix} \text{ by } \begin{matrix} \times 2 \\ \times 3 \end{matrix} \Rightarrow \begin{matrix} 6x + 4y = 14 \\ 6x - 3y = 0 \end{matrix}.$$

Or the number of y's the same:

$$\begin{pmatrix} 3x + 2y = 7 \\ 2x - y = 0 \end{pmatrix} \text{ by } \begin{pmatrix} \times 1 \\ \times 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 3x + 2y = 7 \\ 4x - 2y = 0 \end{pmatrix}.$$

We are then in a position to eliminate (either by adding or subtracting) one of our variables and obtain just a single equation with one variable which we can then solve.

$$\begin{pmatrix} 3x + 2y = 7 \\ 2x - y = 0 \end{pmatrix} \text{ by } \begin{pmatrix} \times 2 \\ \times 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 6x + 4y = 14 \\ 6x - 3y = 0 \end{pmatrix} \ldots\text{so subtract}\ldots(7y = 14)$$

$$\Rightarrow \boxed{y = 2} \Rightarrow (3x + 2 \times 2 = 7) \Rightarrow \boxed{x = 1}.$$

Therefore the solution to the simultaneous equations are $\boxed{x = 1, y = 2}$.

A few worked examples are given below:

- Solve $2x + 3y = -11$ and $-x - 4y = 18$;

$$\begin{pmatrix} 2x + 3y = -11 \\ -x - 4y = 18 \end{pmatrix} \text{ by } \begin{pmatrix} \times 1 \\ \times 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 2x + 3y = -11 \\ -2x - 8y = 36 \end{pmatrix} \ldots\text{so add}\ldots(-5y = 25)$$

$$\Rightarrow \boxed{y = -5} \Rightarrow (2x + 3 \times (-5) = -11) \Rightarrow \boxed{x = 2}.$$

Therefore solution is $\boxed{x = 2, y = -5}$.

- Solve $2x + 3y = 6$ and $3x + y = 1$;

$$\begin{pmatrix} 2x + 3y = 6 \\ 3x + y = 1 \end{pmatrix} \text{ by } \begin{pmatrix} \times 1 \\ \times 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 2x + 3y = 6 \\ 9x + 3y = 3 \end{pmatrix} \ldots\text{so subtract}\ldots(7x = -3)$$

$$\Rightarrow \boxed{x = -\tfrac{3}{7}} \Rightarrow \left(3 \times \left(-\tfrac{3}{7}\right) + y = 1\right) \Rightarrow \boxed{y = 2\tfrac{2}{7}}.$$

Therefore solution is $\boxed{x = -\frac{3}{7}, y = 2\frac{2}{7}}$.

- Solve $ax + by = c$ and $dx + ey = f$ (the most general set of simultaneous equations);

$$\begin{pmatrix} ax + by = c \\ dx + ey = f \end{pmatrix} \text{ by } \begin{pmatrix} \times d \\ \times a \end{pmatrix} \Rightarrow \begin{pmatrix} adx + bdy = cd \\ adx + aey = af \end{pmatrix} \ldots\text{so subtract}\ldots(bdy - aey = cd - af)$$

$$\Rightarrow \boxed{y = \tfrac{cd-af}{bd-ae}} \Rightarrow \left(ax + b \times \left(\tfrac{cd-af}{bd-ae}\right) = c\right) \Rightarrow \boxed{x = \tfrac{bf-ce}{bd-ae}}.$$

Therefore solution is $\boxed{x = \frac{bf-ce}{bd-ae}, y = \frac{cd-af}{bd-ae}}$.
