OCR
Oxford Cambridge and RSA

Wednesday 28 June 2017 – Morning

A2 GCE MATHEMATICS

4731/01 Mechanics 4

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4731/01
- List of Formulae (MF1)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.

- The Question Paper will be found inside the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book**. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Do **not** write in the barcodes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION FOR CANDIDATES

This information is the same on the Printed Answer Book and the Question Paper.

- The number of marks is given in brackets **[ ]** at the end of each question or part question on the Question Paper.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **12** pages. The Question Paper consists of **8** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER/INVIGILATOR

- Do not send this Question Paper for marking; it should be retained in the centre or recycled. Please contact OCR Copyright should you wish to re-use this document.

Answer **all** the questions.

1 A uniform rod with centre C has mass $2M$ and length $4a$. The rod is free to rotate in a vertical plane about a smooth fixed horizontal axis passing through a point A on the rod, where $AC = ka$ and $0 < k < 2$. The rod is making small oscillations about the equilibrium position with period T.

(i) Show that $T = 2\pi\sqrt{\frac{a}{3g}\left(\frac{4+3k^2}{k}\right)}$. (You may assume the standard formula $T = 2\pi\sqrt{\frac{I}{mgh}}$ for the period of small oscillations of a compound pendulum.) **[4]**

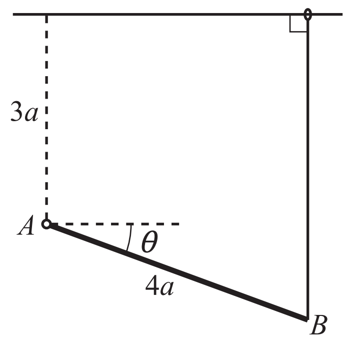
(ii) Hence find the value of k^2 for which the period of oscillations is least. **[3]**

2 A ship S is travelling with constant speed $5\,\mathrm{m\,s^{-1}}$ on a course with bearing 325°. A second ship T observes S when S is 9500 m from T on a bearing of 060° from T. Ship T sets off in pursuit, travelling with constant speed $8.5\,\mathrm{m\,s^{-1}}$ in a straight line.

(i) Find the bearing of the course which T should take in order to intercept S. **[4]**

(ii) Find the distance travelled by S from the moment that T sets off in pursuit until the point of interception. **[5]**

3



A uniform rod AB has mass m and length $4a$. The rod can rotate in a vertical plane about a smooth fixed horizontal axis passing through A. One end of a light elastic string of natural length a and modulus of elasticity λmg is attached to B. The other end of the string is attached to a small light ring which slides on a fixed smooth horizontal rail which is in the same vertical plane as the rod. The rail is a vertical distance $3a$ above A. The string is always vertical and the rod makes an angle θ radians with the horizontal, where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ (see diagram).

(i) Taking A as the reference level for gravitational potential energy, find an expression for the total potential energy V of the system, and show that

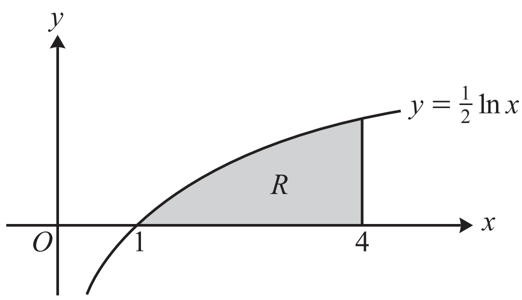
$$\frac{dV}{d\theta} = 2mga\cos\theta\big(4\lambda(1+2\sin\theta)-1\big).$$ **[6]**

Determine the positions of equilibrium and the nature of their stability in the cases

(ii) $\lambda > \frac{1}{12}$, **[9]**

(iii) $\lambda < \frac{1}{12}$. **[2]**

4



The diagram shows the curve with equation $y = \frac{1}{2}\ln x$. The region R, shaded in the diagram, is bounded by the curve, the x-axis and the line $x = 4$. A uniform solid of revolution is formed by rotating R completely about the y-axis to form a solid of volume V.

(i) Show that $V = \frac{1}{4}\pi(64\ln 2 - 15)$. **[4]**

(ii) Find the exact y-coordinate of the centre of mass of the solid. **[7]**

5

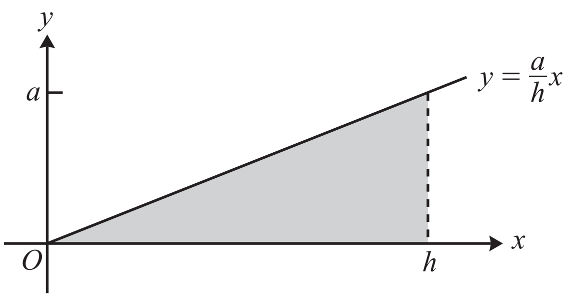


Fig. 1

Fig. 1 shows part of the line $y = \frac{a}{h}x$, where a and h are constants. The shaded region bounded by the line, the x-axis and the line $x = h$ is rotated about the x-axis to form a uniform solid cone of base radius a, height h and volume $\frac{1}{3}\pi a^2 h$. The mass of the cone is M.

(i) Show by integration that the moment of inertia of the cone about the y-axis is $\frac{3}{20}M(a^2 + 4h^2)$. (You may assume the standard formula $\frac{1}{4}mr^2$ for the moment of inertia of a uniform disc about a diameter.) **[7]**

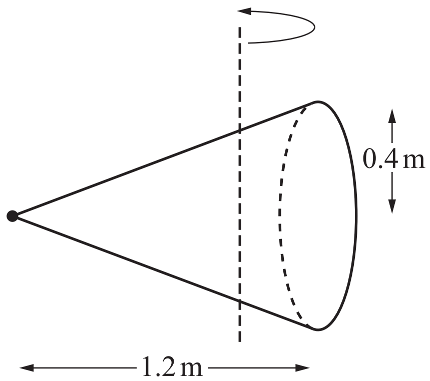
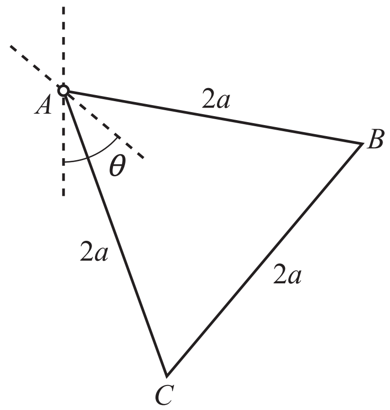


Fig. 2

A uniform solid cone has mass 3 kg, base radius 0.4 m and height 1.2 m. The cone can rotate about a fixed vertical axis passing through its centre of mass with the axis of the cone moving in a horizontal plane. The cone is rotating about this vertical axis at an angular speed of 9.6 rad s^{-1}. A stationary particle of mass m kg becomes attached to the vertex of the cone (see Fig. 2). The particle being attached to the cone causes the angular speed to change instantaneously from 9.6 rad s^{-1} to 7.8 rad s^{-1}.

(ii) Find the value of m. **[5]**

6



A triangular frame ABC consists of three uniform rods AB, BC and CA, rigidly joined at A, B and C. Each rod has mass m and length $2a$. The frame is free to rotate in a vertical plane about a fixed horizontal axis passing through A. The frame is initially held such that the axis of symmetry through A is vertical and BC is below the level of A. The frame starts to rotate with an initial angular speed of ω and at time t the angle between the axis of symmetry through A and the vertical is θ (see diagram).

(i) Show that the moment of inertia of the frame about the axis through A is $6ma^2$. **[3]**

(ii) Show that the angular speed $\dot{\theta}$ of the frame when it has turned through an angle θ satisfies

$$a\dot{\theta}^2 = a\omega^2 - kg\sqrt{3}(1-\cos\theta),$$

stating the exact value of the constant k.

Hence find, in terms of a and g, the set of values of ω^2 for which the frame makes complete revolutions. **[5]**

At an instant when $\theta = \frac{1}{6}\pi$, the force acting on the frame at A has magnitude F.

(iii) Given that $\omega^2 = \dfrac{2g}{a\sqrt{3}}$, find F in terms of m and g. **[8]**

END OF QUESTION PAPER

Question			Answer	Marks	Guidance	
1		(i)	$I=\frac{1}{3}(2M)(2a)^2+2M(ka)^2$	B1 B1	B1 for each term	SC B1: for incorrect masses in both expressions
			$T=2\pi\sqrt{\frac{8Ma^2+6Mk^2a^2}{3(2Mg)(ka)}}$	M1	Using $T=2\pi\sqrt{\frac{I}{mgh}}$ with correct mass	
			$T=2\pi\sqrt{\frac{a}{3g}\left(\frac{4+3k^2}{k}\right)}$	A1 **[4]**	**AG www**	
		(ii)	$\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{4+3k^2}{3k}\right)=0$	M1	Attempt to differentiate $\mathrm{f}(k)$ and set equal to zero	Or differentiates T wrt k
			$\frac{3k(6k)-(4+3k^2)(3)}{(3k)^2}=0 \Rightarrow k^2=...$	M1	Correct application of quotient rule (or other correct method) and get to $k^2=...$ or $k=...$ (if k^2 not stated first)	eg $\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{4}{3k}+k\right)=0$
			$k^2=\frac{4}{3}$	A1 **[3]**	Cao (oe)	
2		(i)	$180-60-35=85$	B1	Correct angle (soi)	
			$\frac{\sin\theta}{5}=\frac{\sin 85}{8.5}$	M1	Sine rule with cv(85)	$\theta=35.873446...$
			bearing = $(180-85-\theta)-35$	M1		
			= 24.1 (3 sf)	A1 **[4]**	24.12655…	
		(ii)	$w^2=5^2+8.5^2-2(5)(8.5)\cos(180-85-\theta)$	*M1	$\frac{w}{\sin(180-85-\theta)}=\frac{8.5}{\sin 85}$	
			$w=7.32$ (3sf)	A1	7.32344…	
			$t=9500/w$	M1dep*		$t=1297.20427$
			$s=5t$	M1	Dependent on both previous M marks	
			$s=6486$ (4sf)	A1 **[5]**	6486.021…	Accept 3sf or better

Question			Answer	Marks	Guidance	
3		(i)	GPE for rod: $-2mga\sin\theta$	B1		
			EPE for string: $\dfrac{\lambda mg(2a+4a\sin\theta)^2}{2a}$	M1 A1	Genuine attempt at extension and substitution into $\dfrac{\lambda x^2}{2a}$	
			$V = 2\lambda mga(1+2\sin\theta)^2 - 2mga\sin\theta$	A1		Accept unsimplified
			$\dfrac{dV}{d\theta} = 4\lambda mga(1+2\sin\theta)(2\cos\theta) - 2mga\cos\theta$	M1	Differentiates their V	
			$\dfrac{dV}{d\theta} = 2mga\cos\theta\left(4\lambda(1+2\sin\theta)-1\right)$	A1 **[6]**	**AG www**	
		(ii)		M1	Set $V' = 0$	
			$\cos\theta = 0 \Rightarrow \theta = \dfrac{\pi}{2}$ and $\sin\theta = \dfrac{1-4\lambda}{8\lambda}\left(\Rightarrow \theta = \sin^{-1}\left(\dfrac{1-4\lambda}{8\lambda}\right)\right)$	A1 A1	A1 for $\theta = \dfrac{\pi}{2}$, A1 for the existence of the root at $\sin^{-1}\left(\dfrac{1-4\lambda}{8\lambda}\right)$ for $\lambda > \frac{1}{12}$	
			$\dfrac{d^2V}{d\theta^2} = -2mga\sin\theta\left(4\lambda(1+2\sin\theta)-1\right)+\ldots$ $\ldots+2mga\cos\theta\left(4\lambda(2\cos\theta)\right)$	M1 A1	M1 Attempt to differentiate V'	At least two terms correct for M mark
			$V'' = -2mga(12\lambda-1)$	M1	Substitute their $\theta = \dfrac{\pi}{2}$ into their V''	
			$\lambda > \dfrac{1}{12} \Rightarrow (12\lambda-1) > 0 \therefore V'' < 0 \Rightarrow$ unstable	A1		
			$V'' = 16mga\lambda\left(1-\left(\dfrac{1-4\lambda}{8\lambda}\right)^2\right)$ or $mga\left(-\dfrac{1}{4\lambda}+2+12\lambda\right)$	M1	Substitute their $\sin\theta = \dfrac{1-4\lambda}{8\lambda}$ into their V'' or $V'' = 16\lambda mga\cos^2\theta$ (which is positive for all values of θ)	
			$\lambda > \dfrac{1}{12} \Rightarrow \left(\dfrac{1-4\lambda}{8\lambda}\right)^2 < 1 \therefore V'' > 0 \Rightarrow$ stable	A1 **[9]**		

Question			Answer	Marks	Guidance	
		(iii)	$V'' = -2mga(12\lambda - 1)$	M1	Substitute $\theta = \frac{\pi}{2}$ (not their θ) into their V''	
			$\lambda < \frac{1}{12} \Rightarrow (12\lambda - 1) < 0 \therefore V'' > 0 \Rightarrow$ stable	A1		
				[2]		

Question			Answer	Marks	Guidance	
4		(i)	$V_1 = \pi\int x^2\mathrm{d}y = \pi\int_0^{\ln 2}\left(\mathrm{e}^{2y}\right)^2\mathrm{d}y$ or $V = \pi\int_0^{\ln 2}\left(4^2 - \left(\mathrm{e}^{2y}\right)^2\right)\mathrm{d}y$	M1	For $\int\left(\mathrm{e}^{2y}\right)^2\mathrm{d}y$	Limits not required for M and first A mark
			$= \pi\left[\frac{1}{4}\mathrm{e}^{4y}\right]_0^{\ln 2}$	A1	For $\frac{1}{4}\mathrm{e}^{4y}$	
			$= \frac{\pi}{4}\left(e^{4\ln 2} - 1\right) = \frac{15\pi}{4}$	A1	For correct substitution of limits	
			$V = \pi(4)^2(\ln 2) - \frac{15\pi}{4} = \frac{1}{4}\pi(64\ln 2 - 15)$	A1	**AG www**	
				[4]		
		(ii)	$V_1\bar{y} = \pi\int yx^2\mathrm{d}y = \pi\int_0^{\ln 2} y\left(\mathrm{e}^{2y}\right)^2\mathrm{d}y = \pi\int_0^{\ln 2} y\mathrm{e}^{4y}\mathrm{d}y$	*M1	For $\int yx^2\mathrm{d}y$	
			or $V\bar{y} = \pi\int 16y - y\mathrm{e}^{4y}\mathrm{d}y$	M1dep*		Clear indication of integrating exponential term and differentiating y term
			$= \pi\left\{\left[\frac{1}{4}y\mathrm{e}^{4y}\right]_0^{\ln 2} - \frac{1}{4}\int_0^{\ln 2}\mathrm{e}^{4y}\mathrm{d}y\right\} = \pi\left[\frac{1}{4}y\mathrm{e}^{4y} - \frac{1}{16}\mathrm{e}^{4y}\right]_0^{\ln 2}$	A2	Both terms integrated correctly (A1 for one error)	Limits not required for M mark and both A marks
			$V_1\bar{y} = \pi\left(\ln 16 - \frac{15}{16}\right)$			
				M1	Table of values idea – dependent on both previous M marks and using exact volume from (i)	
			$\left(\frac{1}{2}\ln 2\right)\left(\pi(4)^2(\ln 2)\right) - \pi\left(\ln 16 - \frac{15}{16}\right) = \bar{y}\left(\pi(4)^2(\ln 2) - \frac{15\pi}{4}\right)$	A1	Two terms correct	
			$\bar{y} = \frac{128(\ln 2)^2 - 16\ln 16 + 15}{256\ln 2 - 60}$	A1	Exact (oe)	For guidance: $\bar{y} = 0.273629\ldots$
				[7]		

Question			Answer	Marks	Guidance	
5		(i)	Mass per unit volume $\frac{3M}{\pi a^2 h}$	B1	oe	
			Moment of inertia of elemental disc about diameter $\frac{1}{4}\left(\rho\pi y^2 \delta x\right) y^2$	B1	$\frac{3Ma^2x^4}{4h^5}\delta x$	
			By the parallel axis theorem, about the given axis	M1*	Condone use of h for x	
			$\frac{1}{4}\rho\pi y^4 \delta x + \left(\rho\pi y^2\delta x\right)x^2$	A1	$\frac{3M}{4h^3}\left(\frac{a^2x^4}{h^2}+4x^4\right)\delta x$	
			$=\frac{1}{4}\rho\pi\frac{a^2}{h^2}\int_0^h \frac{a^2}{h^2}x^4+4x^4 \mathrm{d}x$ or $\frac{3M}{4h^3}\int_0^h\left(\frac{a^2x^4}{h^2}+4x^4\right)\mathrm{d}x$	M1dep*	Condone lack of limits – dependent on previous M mark	
			$=\frac{1}{4}\rho\pi\frac{a^2}{h^2}\left[\frac{a^2x^5}{5h^2}+\frac{4x^5}{5}\right]_0^h$ or $\frac{3M}{4h^3}\left[\frac{a^2x^5}{5h^2}+\frac{4x^5}{5}\right]_0^h$	M1	Integrating and using correct limits – dependent on both previous M marks	
			$=\frac{1}{20}\rho\pi a^2 h\left(a^2+4h^2\right)$			
			$=\frac{1}{20}\left(\frac{3M}{\pi a^2 h}\right)\pi a^2 h\left(a^2+4h^2\right)=\frac{3}{20}M\left(a^2+4h^2\right)$	A1	**AG www**	
				[7]		
		(ii)	$I_1=\frac{3}{20}M\left(a^2+4h^2\right)-M\left(\frac{3}{4}h\right)^2$	M1*	Use of parallel axis theorem to find moment of inertia through the CoM	Allow + for the M mark
			$I_1=\frac{3}{80}(3)\left(4(0.4)^2+(1.2)^2\right)$	A1	$I_1=\frac{3M}{80}\left(4a^2+h^2\right)$	0.234
			$(I_2=)I_1+m\left(0.75(1.2)\right)^2$	B1	$I_1 + 0.81m$ allow for any I_1	
			$9.6I_1=7.8I_2$	M1dep*	Using cons. of angular momentum	
			$m=\frac{1}{15}$	A1	0.0666…	
				[5]		

Question			Answer	Marks	Guidance	
6		(i)	Moment of inertia of rods AB and AC: $2\left(\frac{4}{3}ma^2\right)$	B1		
			Moment of inertia of BC about A: $\frac{1}{3}ma^2+3ma^2$	M1	Use of parallel axis theorem for M of I of BC about A	
			$I=\frac{8}{3}ma^2+\frac{10}{3}ma^2=6ma^2$	A1	**AG www**	
				[3]		
		(ii)		M1	Equation involving KE (must involve I and two terms) and PE (two terms)	
			$\frac{1}{2}\left(6ma^2\right)\dot{\theta}^2-3mg\left(\frac{2}{3}\sqrt{3}a\cos\theta\right)=\frac{1}{2}\left(6ma^2\right)\omega^2-3mg\left(\frac{2}{3}\sqrt{3}a\right)$	A1	Or B1 for either the KE or PE terms correct	
			$a\dot{\theta}^2=\frac{2}{3}\omega^2-kg\sqrt{3}(1-\cos\theta)$	A1	$k=\frac{2}{3}$	
				M1	Setting $\theta=\pi$ and $\dot{\theta}>0$	Condone ≥ 0 or $=0$ for the M mark
			$\omega^2>\frac{4g}{a\sqrt{3}}$	A1		
				[5]		

Question			Answer	Marks	Guidance	
		(iii)		M1	Attempt to differentiate their $\dot{\theta}$ wrt to t	
			$\ddot{\theta} = -\frac{kg\sqrt{3}}{2a}\sin\theta$ or $-\frac{kg\sqrt{3}}{4a}$	A1ft	Follow through their k (or allow in terms of k)	
			$Y - 3mg\cos\theta = 3mh\dot{\theta}^2$	*M1	For radial acceleration $r\omega^2$ - must sub for $\dot{\theta}^2$ - allow incorrect m and r for M mark only	
			$Y = 3mg\cos\theta + 3m\left(\frac{2\sqrt{3}a}{3}\right)\left(\frac{2g\sqrt{3}\cos\theta}{3a}\right)$	A1	$Y = \frac{7\sqrt{3}}{2}mg$	
			$X - 3mg\sin\theta = 3mh\ddot{\theta}$	*M1	For transverse acceleration $r\alpha$ - must sub their α - allow incorrect m and r for M mark only	
			$X = 3mg\sin\theta + 3m\left(\frac{2\sqrt{3}a}{3}\right)\left(-\frac{2g\sqrt{3}\sin\theta}{6a}\right)$	A1	$X = \frac{1}{2}mg$	
			$F = \sqrt{X^2 + Y^2} = \sqrt{\left(\frac{7\sqrt{3}mg}{2}\right)^2 + \left(\frac{mg}{2}\right)^2}$	M1 dep*	Substituting $\theta = \frac{\pi}{6}$ into X and Y and applying formula for F. This substitution could be done initially – must be using correct m and r	
			$F = mg\sqrt{37}$	A1 **[8]**		