OCR

RECOGNISING ACHIEVEMENT

ADVANCED GCE UNIT

4731/01

MATHEMATICS

Mechanics 4

FRIDAY 22 JUNE 2007

Morning

Time: 1 hour 30 minutes

Additional Materials: Answer Booklet (8 pages)
List of Formulae (MF1)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.

ADVICE TO CANDIDATES

- Read each question carefully and make sure you know what you have to do before starting your answer.
- **You are reminded of the need for clear presentation in your answers.**

This document consists of **6** printed pages and **2** blank pages.

[Turn over

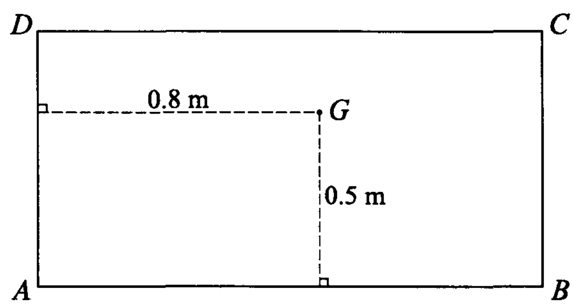
1 The driveshaft of an electric motor begins to rotate from rest and has constant angular acceleration. In the first 8 seconds it turns through 56 radians.

(i) Find the angular acceleration. [2]

(ii) Find the angle through which the driveshaft turns while its angular speed increases from 20 rad s^{-1} to 36 rad s^{-1}. [2]

2 The region R is bounded by the curve $y = \sqrt{4a^2 - x^2}$ for $0 \leqslant x \leqslant a$, the x-axis, the y-axis and the line $x = a$, where a is a positive constant. The region R is rotated through 2π radians about the x-axis to form a uniform solid of revolution. Find the x-coordinate of the centre of mass of this solid. [7]

3



A non-uniform rectangular lamina $ABCD$ has mass 6 kg. The centre of mass G of the lamina is 0.8 m from the side AD and 0.5 m from the side AB (see diagram). The moment of inertia of the lamina about AD is 6.2 kg m^2 and the moment of inertia of the lamina about AB is 2.8 kg m^2.

The lamina rotates in a vertical plane about a fixed horizontal axis which passes through A and is perpendicular to the lamina.

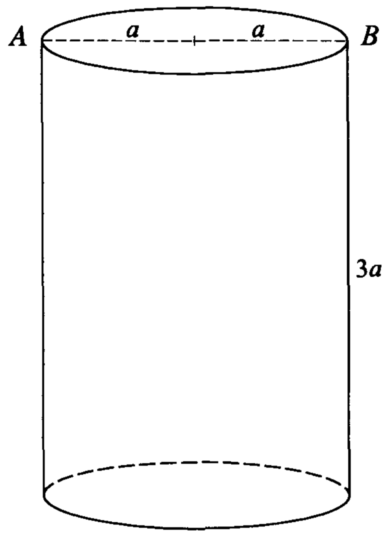
(i) Write down the moment of inertia of the lamina about this axis. [1]

The lamina is released from rest in the position where AB and DC are horizontal and DC is above AB. A frictional couple of constant moment opposes the motion. When AB is first vertical, the angular speed of the lamina is 2.4 rad s^{-1}.

(ii) Find the moment of the frictional couple. [5]

(iii) Find the angular acceleration of the lamina immediately after it is released. [3]

4



A uniform solid cylinder has radius a, height $3a$, and mass M. The line AB is a diameter of one of the end faces of the cylinder (see diagram).

(i) Show by integration that the moment of inertia of the cylinder about AB is $\frac{13}{4}Ma^2$. (You may assume that the moment of inertia of a uniform disc of mass m and radius a about a diameter is $\frac{1}{4}ma^2$.) [7]

The line AB is now fixed in a horizontal position and the cylinder rotates freely about AB, making small oscillations as a compound pendulum.

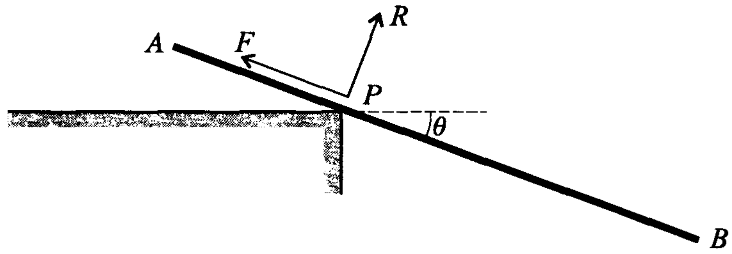
(ii) Find the approximate period of these small oscillations, in terms of a and g. [3]

5 A ship S is travelling with constant speed 12 m s^{-1} on a course with bearing 345°. A patrol boat B spots the ship S when S is 2400 m from B on a bearing of 050°. The boat B sets off in pursuit, travelling with constant speed $v\text{ m s}^{-1}$ in a straight line.

(i) Given that $v = 16$, find the bearing of the course which B should take in order to intercept S, and the time taken to make the interception. [8]

(ii) Given instead that $v = 10$, find the bearing of the course which B should take in order to get as close as possible to S. [4]

6



A uniform rod AB has mass m and length $2a$. The point P on the rod is such that $AP = \frac{2}{3}a$. The rod is placed in a horizontal position perpendicular to the edge of a rough horizontal table, with AP in contact with the table and PB overhanging the edge. The rod is released from rest in this position. When it has rotated through an angle θ, and no slipping has occurred at P, the normal reaction acting on the rod at P is R and the frictional force is F (see diagram).

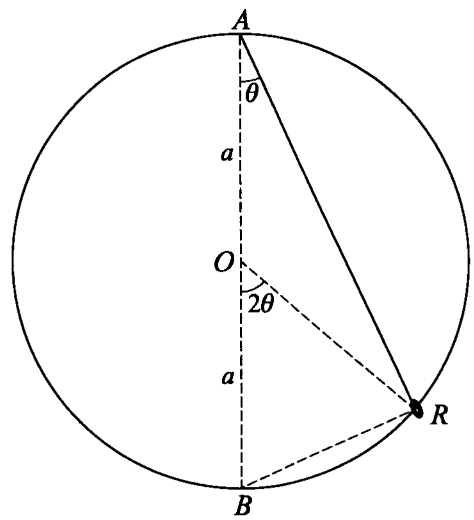
(i) Show that the angular acceleration of the rod is $\dfrac{3g\cos\theta}{4a}$. [4]

(ii) Find the angular speed of the rod, in terms of a, g and θ. [3]

(iii) Find F and R in terms of m, g and θ. [6]

(iv) Given that the coefficient of friction between the rod and the edge of the table is μ, show that the rod is on the point of slipping at P when $\tan\theta = \frac{1}{2}\mu$. [2]

7



A smooth circular wire, with centre O and radius a, is fixed in a vertical plane. The highest point on the wire is A and the lowest point on the wire is B. A small ring R of mass m moves freely along the wire. A light elastic string, with natural length a and modulus of elasticity $\frac{1}{2}mg$, has one end attached to A and the other end attached to R. The string AR makes an angle θ (measured anticlockwise) with the downward vertical, so that OR makes an angle 2θ with the downward vertical (see diagram). You may assume that the string does not become slack.

(i) Taking A as the level for zero gravitational potential energy, show that the total potential energy V of the system is given by

$$V = mga\left(\tfrac{1}{4} - \cos\theta - \cos^2\theta\right).$$ [4]

(ii) Show that $\theta = 0$ is the only position of equilibrium. [3]

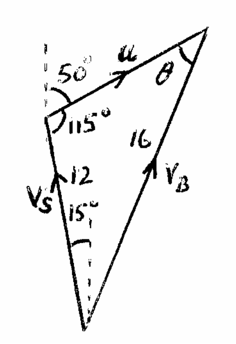
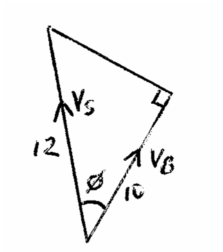
(iii) By differentiating the energy equation with respect to time t, show that

$$\frac{d^2\theta}{dt^2} = -\frac{g}{4a}\sin\theta(1 + 2\cos\theta).$$ [5]

(iv) Deduce the approximate period of small oscillations about the equilibrium position $\theta = 0$. [3]

1 (i)	Using $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$, $56 = 0 + \frac{1}{2}\alpha \times 8^2$ $\alpha = 1.75 \text{ rad s}^{-2}$	M1 A1 **2**	
(ii)	Using $\omega_1^2 = \omega_0^2 + 2\alpha\theta$, $36^2 = 20^2 + 2 \times 1.75\theta$ $\theta = 256$ rad	M1 A1 ft **2**	ft is $448 \div \alpha$
2	Volume is $\int_0^a \pi(4a^2 - x^2)\,dx = \pi\left[4a^2x - \frac{1}{3}x^3\right]_0^a$ $= \frac{11}{3}\pi a^3$ $\int_0^a \pi x(4a^2 - x^2)\,dx$ $= \pi\left[2a^2x^2 - \frac{1}{4}x^4\right]_0^a$ $= \frac{7}{4}\pi a^4$ $\bar{x} = \frac{\frac{7}{4}\pi a^4}{\frac{11}{3}\pi a^3}$ $= \frac{21}{44}a$	M1 A1 M1 A1 A1 M1 A1 **7**	*π may be omitted throughout* (Limits not required) (Limits not required) for $\frac{\int xy^2\,dx}{\int y^2\,dx}$
3 (i)	$I = 6.2 + 2.8 = 9.0 \text{ kg m}^2$	B1 **1**	
(ii)	WD against frictional couple is $L \times \frac{1}{2}\pi$ Loss of PE is $6 \times 9.8 \times 1.3$ $(= 76.44)$ Gain of KE is $\frac{1}{2} \times 9.0 \times 2.4^2$ $(= 25.92)$ By work-energy principle, $L \times \frac{1}{2}\pi = 76.44 - 25.92$ $L = 32.2 \text{ N m}$	B1 B1 B1 ft M1 A1 **5**	Equation involving WD, KE and PE Accept 32.1 to 32.2
(iii)	$6 \times 9.8 \times 0.8 - L = I\alpha$ $\alpha = 1.65 \text{ rad s}^{-2}$	M1 A1 ft A1 **3**	Moments equation

4 (i)	MI of elemental disc about a diameter is $\frac{1}{4}\left(\frac{M}{3a}\delta x\right)a^2$	B1	$\frac{M}{3a}$ may be $\rho\pi a^2$ throughout *(condone use of $\rho = 1$)*
	MI of elemental disc about *AB* is $\frac{1}{4}\left(\frac{M}{3a}\delta x\right)a^2 + \left(\frac{M}{3a}\delta x\right)x^2$	M1 A1	Using parallel axes rule (can award A1 for $\frac{1}{4}ma^2 + mx^2$)
	$I = \frac{M}{3a}\int_0^{3a}(\frac{1}{4}a^2 + x^2)\,\mathrm{d}x$	M1 A1	Integrating MI of disc *about AB* Correct integral expression for *I*
	$= \frac{M}{3a}\left[\frac{1}{4}a^2x + \frac{1}{3}x^3\right]_0^{3a}$ $= \frac{M}{3a}(\frac{3}{4}a^3 + 9a^3)$ $= M(\frac{1}{4}a^2 + 3a^2)$	M1	Obtaining an expression for *I* in terms of *M* and *a* *Dependent on previous M1*
	$= \frac{13}{4}Ma^2$	A1 (ag) **7**	
(ii)	Period is $2\pi\sqrt{\frac{I}{Mgh}}$	M1	or $-Mgh\sin\theta = I\ddot{\theta}$
	$= 2\pi\sqrt{\frac{\frac{13}{4}Ma^2}{Mg\frac{3}{2}a}}$	A1	
	$= 2\pi\sqrt{\frac{13a}{6g}}$	A1 **3**	

5 (i)	 $\frac{\sin\theta}{12} = \frac{\sin 115}{16}$ $\theta = 42.8°$ Bearing of $\mathbf{v}_B$ is 007.2° $\frac{u}{\sin 22.2} = \frac{16}{\sin 115}$ $u = 6.66$ Time taken is $\frac{2400}{6.664} = 360$ s	M1 A1 M1 A1 M1 A1 M1*A1 ft **8**	Relative velocity on bearing 050 Correct velocity diagram; or $\begin{pmatrix} u\sin 50 \\ u\cos 50 \end{pmatrix} = \begin{pmatrix} 16\sin\alpha \\ 16\cos\alpha \end{pmatrix} - \begin{pmatrix} 12\sin 345 \\ 12\cos 345 \end{pmatrix}$ or eliminating u (or α) or obtaining equation for u (or α) *For equations in α and t* *M1*M1A1 for equations* *M1 for eliminating t (or α)* *A1 for $\alpha = 7.2$* *M1A1 ft for equation for t (or α)* *A1 cao for $t = 360$*
(ii)	 $\cos\phi = \frac{10}{12}$ $\phi = 33.6°$ Bearing of $\mathbf{v}_B$ is 018.6°	M1 A1 M1 A1 **4**	Relative velocity perpendicular to $\mathbf{v}_B$ Correct velocity diagram For alternative methods: M2 for a completely correct method A2 for 018.6 (give A1 for a correct relevant angle)

6 (i)	$I = \frac{1}{3}ma^2 + m(\frac{1}{3}a)^2$	M1	Using parallel axes rule
	$= \frac{4}{9}ma^2$	A1	
	$mg(\frac{1}{3}a\cos\theta) = I\alpha$	M1	
	$\alpha = \dfrac{\frac{1}{3}mga\cos\theta}{\frac{4}{9}ma^2} = \dfrac{3g\cos\theta}{4a}$	A1 (ag) **4**	
(ii)	By conservation of energy,	M1	
	$\frac{1}{2}I\omega^2 = mg(\frac{1}{3}a\sin\theta)$	A1 ft	
	$\frac{2}{9}ma^2\omega^2 = \frac{1}{3}mga\sin\theta$		
	$\omega = \sqrt{\dfrac{3g\sin\theta}{2a}}$	A1 **3**	*Condone* $\omega^2 = \dfrac{3g\sin\theta}{2a}$
	OR $\omega\dfrac{d\omega}{d\theta} = \dfrac{3g\cos\theta}{4a}$ M1		
	$\frac{1}{2}\omega^2 = \int \dfrac{3g\cos\theta}{4a}d\theta$		
	$= \dfrac{3g\sin\theta}{4a}$ $(+C)$ A1		
	$\omega = \sqrt{\dfrac{3g\sin\theta}{2a}}$ A1		
(iii)	Acceleration parallel to rod is $(\frac{1}{3}a)\omega^2$	B1	
	$F - mg\sin\theta = m(\frac{1}{3}a)\omega^2$	M1	Radial equation with 3 terms
	$F - mg\sin\theta = \frac{1}{2}mg\sin\theta$		
	$F = \frac{3}{2}mg\sin\theta$	A1	
	Acceleration perpendicular to rod is $(\frac{1}{3}a)\alpha$	B1 ft	ft is $r\alpha$ with r the same as before
	$mg\cos\theta - R = m(\frac{1}{3}a)\alpha$	M1	Transverse equation with 3 terms
	$mg\cos\theta - R = \frac{1}{4}mg\cos\theta$		
	$R = \frac{3}{4}mg\cos\theta$	A1 **6**	
	OR $R(\frac{1}{3}a) = I_G\alpha$ M1		Must use I_G
	$R(\frac{1}{3}a) = (\frac{1}{3}ma^2)\left(\dfrac{3g\cos\theta}{4a}\right)$ A1		
	$R = \frac{3}{4}mg\cos\theta$ A1		
(iv)	On the point of slipping, $F = \mu R$		
	$\frac{3}{2}mg\sin\theta = \mu(\frac{3}{4}mg\cos\theta)$	M1	
	$\tan\theta = \frac{1}{2}\mu$	A1 (ag) **2**	Correctly obtained *Dependent on 6 marks earned in (iii)*

7 (i)	GPE $= (-)\, mg(2a\cos\theta)\cos\theta$ EPE $= \dfrac{\frac{1}{2}mg}{2a}(AR-a)^2$ $= \dfrac{\frac{1}{2}mg}{2a}(2a\cos\theta - a)^2$ $V = \frac{1}{4}mga(2\cos\theta-1)^2 - 2mga\cos^2\theta$ $= mga(\cos^2\theta - \cos\theta + \frac{1}{4} - 2\cos^2\theta)$ $= mga(\frac{1}{4} - \cos\theta - \cos^2\theta)$	B1 M1 A1 A1 (ag) **4**	or $(-)\, mg(a + a\cos 2\theta)$
(ii)	$\dfrac{dV}{d\theta} = mga(\sin\theta + 2\cos\theta\sin\theta)$ $= mga\sin\theta(1+2\cos\theta)$ Equilibrium when $\dfrac{dV}{d\theta} = 0$ ie when $\theta = 0$	B1 M1 A1 (ag) **3**	
(iii)	KE is $\frac{1}{2}m(2a\dot{\theta})^2$ $2ma^2\dot{\theta}^2 + V = \text{constant}$ Differentiating with respect to t, $4ma^2\dot{\theta}\ddot{\theta} + \dfrac{dV}{d\theta}\dot{\theta} = 0$ $4ma^2\dot{\theta}\ddot{\theta} + mga\sin\theta(1+2\cos\theta)\dot{\theta} = 0$ $\ddot{\theta} = -\dfrac{g}{4a}\sin\theta(1+2\cos\theta)$	B1 M1 M1 A1 ft A1 (ag) **5**	(can award this M1 if no KE term) *SR* B2 (replacing the last 3 marks) for the given result correctly obtained by differentiating w.r.t. θ
(iv)	When θ is small, $\sin\theta \approx \theta$, $\cos\theta \approx 1$ $\ddot{\theta} \approx -\dfrac{g}{4a}\theta(1+2) = -\dfrac{3g}{4a}\theta$ Period is $2\pi\sqrt{\dfrac{4a}{3g}}$	M1 A1 A1 **3**	