OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS

4727

Further Pure Mathematics 3

Candidates answer on the Answer Booklet

OCR Supplied Materials:
- 8 page Answer Booklet
- List of Formulae (MF1)

Other Materials Required:
None

Friday 29 January 2010
Morning

Duration: 1 hour 30 minutes

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INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

Turn over

1 Determine whether the lines

$$\frac{x-1}{1} = \frac{y+2}{-1} = \frac{z+4}{2} \qquad \text{and} \qquad \frac{x+3}{2} = \frac{y-1}{3} = \frac{z-5}{4}$$

intersect or are skew. **[5]**

2 H denotes the set of numbers of the form $a + b\sqrt{5}$, where a and b are rational. The numbers are combined under multiplication.

(i) Show that the product of any two members of H is a member of H. **[2]**

It is now given that, for a and b not both zero, H forms a group under multiplication.

(ii) State the identity element of the group. **[1]**

(iii) Find the inverse of $a + b\sqrt{5}$. **[2]**

(iv) With reference to your answer to part **(iii)**, state a property of the number 5 which ensures that every number in the group has an inverse. **[1]**

3 Use the integrating factor method to find the solution of the differential equation

$$\frac{dy}{dx} + 2y = e^{-3x}$$

for which $y = 1$ when $x = 0$. Express your answer in the form $y = f(x)$. **[6]**

4 **(i)** Write down, in cartesian form, the roots of the equation $z^4 = 16$. **[2]**

(ii) Hence solve the equation $w^4 = 16(1 - w)^4$, giving your answers in cartesian form. **[5]**

5 A regular tetrahedron has vertices at the points

$$A\left(0, 0, \tfrac{2}{3}\sqrt{6}\right), \quad B\left(\tfrac{2}{3}\sqrt{3}, 0, 0\right), \quad C\left(-\tfrac{1}{3}\sqrt{3}, 1, 0\right), \quad D\left(-\tfrac{1}{3}\sqrt{3}, -1, 0\right).$$

(i) Obtain the equation of the face ABC in the form

$$x + \sqrt{3}y + \left(\tfrac{1}{2}\sqrt{2}\right)z = \tfrac{2}{3}\sqrt{3}.$$ **[5]**

(Answers which only verify the given equation will not receive full credit.)

(ii) Give a geometrical reason why the equation of the face ABD can be expressed as

$$x - \sqrt{3}y + \left(\tfrac{1}{2}\sqrt{2}\right)z = \tfrac{2}{3}\sqrt{3}.$$ **[2]**

(iii) Hence find the cosine of the angle between two faces of the tetrahedron. **[4]**

6 The variables x and y satisfy the differential equation

$$\frac{d^2y}{dx^2} + 16y = 8\cos 4x.$$

(i) Find the complementary function of the differential equation. **[2]**

(ii) Given that there is a particular integral of the form $y = px\sin 4x$, where p is a constant, find the general solution of the equation. **[6]**

(iii) Find the solution of the equation for which $y = 2$ and $\frac{dy}{dx} = 0$ when $x = 0$. **[4]**

7 **(i)** Solve the equation $\cos 6\theta = 0$, for $0 < \theta < \pi$. **[3]**

(ii) By using de Moivre's theorem, show that

$$\cos 6\theta \equiv (2\cos^2\theta - 1)(16\cos^4\theta - 16\cos^2\theta + 1).$$ **[5]**

(iii) Hence find the exact value of

$$\cos\left(\tfrac{1}{12}\pi\right)\cos\left(\tfrac{5}{12}\pi\right)\cos\left(\tfrac{7}{12}\pi\right)\cos\left(\tfrac{11}{12}\pi\right),$$

justifying your answer. **[5]**

8 The function f is defined by $\mathrm{f} : x \mapsto \frac{1}{2-2x}$ for $x \in \mathbb{R}$, $x \neq 0$, $x \neq \frac{1}{2}$, $x \neq 1$. The function g is defined by $\mathrm{g}(x) = \mathrm{ff}(x)$.

(i) Show that $\mathrm{g}(x) = \frac{1-x}{1-2x}$ and that $\mathrm{gg}(x) = x$. **[4]**

It is given that f and g are elements of a group K under the operation of composition of functions. The element e is the identity, where $\mathrm{e} : x \mapsto x$ for $x \in \mathbb{R}$, $x \neq 0$, $x \neq \frac{1}{2}$, $x \neq 1$.

(ii) State the orders of the elements f and g. **[2]**

(iii) The inverse of the element f is denoted by h. Find $\mathrm{h}(x)$. **[2]**

(iv) Construct the operation table for the elements e, f, g, h of the group K. **[4]**

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1	METHOD 1			
	line segment between l_1 and l_2 = $\pm[4, -3, -9]$	B1		For correct vector
	$\mathbf{n} = [1, -1, 2] \times [2, 3, 4] = (\pm)[-2, 0, 1]$	M1* A1		For finding vector product of direction vectors
	distance = $\dfrac{\lvert[4, -3, -9] \cdot [-2, 0, 1]\rvert}{\left(\sqrt{2^2+0^2+1^2}\right)} = \dfrac{17}{\left(\sqrt{5}\right)}$	M1 (*dep)		For using numerator of distance formula
	$\neq 0$, so skew	A1	5	For correct scalar product and correct conclusion
	METHOD 2 lines would intersect where			
	$\left.\begin{array}{l} 1+s=-3+2t \\ -2-s=1+3t \\ -4+2s=5+4t \end{array}\right\} \Rightarrow \left\{\begin{array}{l} s-2t=-4 \\ s+3t=-3 \\ 2s-4t=9 \end{array}\right.$	B1		For correct parametric form for either line
		M1*		For 3 equations using 2 different parameters
		A1		
		M1 (*dep)		For attempting to solve to show (in)consistency
	$\Rightarrow$ contradiction, so skew	A1		For correct conclusion
			5	
2 (i)	$\left(a+b\sqrt{5}\right)\left(c+d\sqrt{5}\right)$	M1		For using product of 2 distinct elements
	$= ac+5bd+(bc+ad)\sqrt{5} \in H$	A1	2	For correct expression
(ii)	$(e=)\ 1$ *OR* $1+0\sqrt{5}$	B1	1	For correct identity
(iii)	*EITHER* $\dfrac{1}{a+b\sqrt{5}} \times \dfrac{a-b\sqrt{5}}{a-b\sqrt{5}}$	M1		For correct inverse as $\left(a+b\sqrt{5}\right)^{-1}$ and multiplying top and bottom by $a-b\sqrt{5}$
	OR $\left(a+b\sqrt{5}\right)\left(c+d\sqrt{5}\right)=1 \Rightarrow \left\{\begin{array}{l} ac+5bd=1 \\ bc+ad=0 \end{array}\right.$			*OR* for using definition and equating parts
	inverse $= \dfrac{a}{a^2-5b^2} - \dfrac{b}{a^2-5b^2}\sqrt{5}$	A1	2	For correct inverse. Allow as a single fraction
(iv)	5 is prime *OR* $\sqrt{5} \notin \mathbb{Q}$	B1	1	For a correct property (or equivalent)
			6	
3	Integrating factor = $e^{\int 2\,dx} = e^{2x}$	B1		For correct IF
	$\Rightarrow \dfrac{d}{dx}\left(ye^{2x}\right) = e^{-x}$	M1		For $\dfrac{d}{dx}(y.\text{their IF}) = e^{-3x}$. their IF
	$\Rightarrow y e^{2x} = -e^{-x} (+c)$	A1		For correct integration both sides
	$(0, 1) \Rightarrow c = 2$	M1		For substituting $(0, 1)$ into their GS and solving for c
		A1√		For correct c f.t. from their GS
	$\Rightarrow y = -e^{-3x} + 2e^{-2x}$	A1	6	For correct solution
			6	
4 (i)	$(z=)\ 2, -2, 2i, -2i$	M1		For at least 2 roots of the form $k\{1, i\}$ **AEF**
		A1	2	For correct values

(ii)	$\frac{w}{1-w} = 2, -2, 2\mathrm{i}, -2\mathrm{i}$	M1	For $\frac{w}{1-w}$ = any one solution from **(i)**
	$w = \frac{z}{1+z}$	M1	For attempting to solve for w, using any solution or in general
	$w = \frac{2}{3}, 2$	B1	For any one of the 4 solutions
		A1	For both real solutions
	$w = \frac{4}{5} \pm \frac{2}{5}\mathrm{i}$	A1 **5**	For both complex solutions **SR** Allow B1√ and one A1√ from $k \neq 2$
		7	
5 (i)	$\mathbf{AB} = k\left[\frac{2}{3}\sqrt{3}, 0, -\frac{2}{3}\sqrt{6}\right]$,	B1	For any one edge vector of ΔABC
	$\mathbf{BC} = k\left[-\sqrt{3}, 1, 0\right]$, $\mathbf{CA} = k\left[\frac{1}{3}\sqrt{3}, -1, \frac{2}{3}\sqrt{6}\right]$	B1	For any other edge vector of ΔABC
	$\mathbf{n} = k_1\left[\frac{2}{3}\sqrt{6}, \frac{2}{3}\sqrt{18}, \frac{2}{3}\sqrt{3}\right] = k_2\left[1, \sqrt{3}, \frac{1}{2}\sqrt{2}\right]$	M1	For attempting to find vector product of any two edges
		M1	For substituting A, B or C into $\mathbf{r}.\mathbf{n}$
	substitute A, B or C $\Rightarrow x + \sqrt{3}y + \frac{1}{2}\sqrt{2}z = \frac{2}{3}\sqrt{3}$	A1 **5**	For correct equation **AG**
			SR For verification only allow M1, then A1 for 2 points and A1 for the third point
(ii)	Symmetry	B1*	For quoting symmetry or reflection
	in plane OAB or Oxz or $y = 0$	B1 (*dep)**2**	For correct plane Allow "in y coordinates" or "in y axis" **SR** For symmetry implied by reference to opposite signs in y coordinates of C and D, award B1 only
(iii)	$\cos\theta = \frac{\left\lvert\left[1, \sqrt{3}, \frac{1}{2}\sqrt{2}\right].\left[1, -\sqrt{3}, \frac{1}{2}\sqrt{2}\right]\right\rvert}{\sqrt{1+3+\frac{1}{2}}\sqrt{1+3+\frac{1}{2}}}$	M1	For using scalar product of normal vectors
		A1	For correct scalar product
	$= \frac{\left\lvert 1-3+\frac{1}{2}\right\rvert}{\frac{9}{2}} = \frac{\frac{3}{2}}{\frac{9}{2}} = \frac{1}{3}$	M1	For product of both moduli in denominator
		A1 **4**	For correct answer. Allow $-\frac{1}{3}$
		11	
6 (i)	$\left(m^2 + 16 = 0 \Rightarrow\right)$ $m = \pm 4\mathrm{i}$	M1	For attempt to solve correct auxiliary equation (may be implied by correct CF)
	CF = $A\cos 4x + B\sin 4x$	A1 **2**	For correct CF (**AEtrig** but not $A\mathrm{e}^{4\mathrm{i}x} + B\mathrm{e}^{-4\mathrm{i}x}$ only)
(ii)	$\frac{\mathrm{d}y}{\mathrm{d}x} = p\sin 4x + 4px\cos 4x$	M1	For differentiating PI twice, using product rule
		A1	For correct $\frac{\mathrm{d}y}{\mathrm{d}x}$
	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 8p\cos 4x - 16px\sin 4x$	A1√	For unsimplified $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. f.t. from $\frac{\mathrm{d}y}{\mathrm{d}x}$
	$\Rightarrow 8p\cos 4x = 8\cos 4x$	M1	For substituting into DE
	$\Rightarrow p = 1$	A1	For correct p
	$\Rightarrow (y =) A\cos 4x + B\sin 4x + x\sin 4x$	B1√ **6**	For using GS = CF + PI, with 2 arbitrary constants in CF and none in PI

(iii)	$(0, 2) \Rightarrow A = 2$	B1√	For correct A. f.t. from their GS
	$\frac{dy}{dx} = -4A\sin 4x + 4B\cos 4x + \sin 4x + 4x\cos 4x$	M1	For differentiating their GS
	$x = 0,\ \frac{dy}{dx} = 0 \Rightarrow B = 0$	M1	For substituting values for x and $\frac{dy}{dx}$ to find B
	$\Rightarrow y = 2\cos 4x + x\sin 4x$	A1 **4**	For stating correct solution **CAO** including $\boxed{y=}$
		$\boxed{12}$	
7 (i)	$\cos 6\theta = 0 \Rightarrow 6\theta = k \times \frac{1}{2}\pi$	M1	For multiples of $\frac{1}{2}\pi$ seen or implied
	$\Rightarrow \theta = \frac{1}{12}\pi\{1, 3, 5, 7, 9, 11\}$	A1	A1 for any 3 correct
		A1 **3**	A1 for the rest, and no extras in $0 < \theta < \pi$
(ii)	METHOD 1		
	$\text{Re}(c + \text{i}s)^6 = \cos 6\theta = c^6 - 15c^4s^2 + 15c^2s^4 - s^6$	M1	For expanding $(c + \text{i}s)^6$ at least 4 terms and 2 binomial coefficients needed
		A1	For 4 correct terms
	$\cos 6\theta = c^6 - 15c^4(1-c^2) + 15c^2(1-c^2)^2 - (1-c^2)^3$	M1	For using $s^2 = 1 - c^2$
	$\Rightarrow \cos 6\theta = 32c^6 - 48c^4 + 18c^2 - 1$	A1	For correct expression for $\cos 6\theta$
	$\Rightarrow \cos 6\theta = \left(2c^2 - 1\right)\left(16c^4 - 16c^2 + 1\right)$	A1 **5**	For correct result **AG** (may be written down from correct $\cos 6\theta$)
	METHOD 2		
	$\text{Re}(c + \text{i}s)^3 = \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$	M1	For expanding $(c + \text{i}s)^3$ at least 2 terms and 1 binomial coefficient needed
		A1	For 2 correct terms
	$\Rightarrow \cos 6\theta = \cos 2\theta\left(\cos^2 2\theta - 3\sin^2 2\theta\right)$	M1	For replacing θ by 2θ
	$\Rightarrow \cos 6\theta = \left(2\cos^2\theta - 1\right)\left(4\left(2\cos^2\theta - 1\right)^2 - 3\right)$	A1	For correct expression in $\cos\theta$ (unsimplified)
	$\Rightarrow \cos 6\theta = \left(2c^2 - 1\right)\left(16c^4 - 16c^2 + 1\right)$	A1	For correct result **AG**
(iii)	METHOD 1		
	$\cos 6\theta = 0$	M1	For putting $\cos 6\theta = 0$
	$\Rightarrow$ 6 roots of $\cos 6\theta = 0$ satisfy $16c^4 - 16c^2 + 1 = 0$ and $2c^2 - 1 = 0$	A1	For association of roots with quartic and quadratic
	But $\theta = \frac{1}{4}\pi, \frac{3}{4}\pi$ satisfy $2c^2 - 1 = 0$	B1	For correct association of roots with quadratic
	EITHER Product of 4 roots *OR* $c = \pm\frac{1}{2}\sqrt{2 \pm \sqrt{3}}$	M1	For using product of 4 roots *OR* for solving quartic
	$\Rightarrow \cos\frac{1}{12}\pi \cos\frac{5}{12}\pi \cos\frac{7}{12}\pi \cos\frac{11}{12}\pi = \frac{1}{16}$	A1 **5**	For correct value (may follow A0 and B0)

	METHOD 2			
	$\cos 6\theta = 0$	M1		For putting $\cos 6\theta = 0$
	$\Rightarrow$ 6 roots of $\cos 6\theta = 0$ satisfy $32c^6 - 48c^4 + 18c^2 - 1 = 0$	A1		For association of roots with sextic
	Product of 6 roots $\Rightarrow$	M1		For using product of 6 roots
	$\cos\frac{1}{12}\pi \cdot \frac{1}{\sqrt{2}} \cdot \cos\frac{5}{12}\pi \cos\frac{7}{12}\pi \cdot \frac{-1}{\sqrt{2}} \cdot \cos\frac{11}{12}\pi = -\frac{1}{32}$	B1		For using $\cos\left\{\frac{3}{12}\pi, \frac{9}{12}\pi\right\} = \left\{\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right\}$
	$\cos\frac{1}{12}\pi \cos\frac{5}{12}\pi \cos\frac{7}{12}\pi \cos\frac{11}{12}\pi = \frac{1}{16}$	A1		For correct value
			13	
8 (i)	$g(x) = \dfrac{1}{2 - 2 \cdot \dfrac{1}{2-2x}} = \dfrac{2-2x}{2-4x} = \dfrac{1-x}{1-2x}$	M1		For use of $ff(x)$
		A1		For correct expression **AG**
	$gg(x) = \dfrac{1 - \dfrac{1-x}{1-2x}}{1 - 2 \cdot \dfrac{1-x}{1-2x}} = \dfrac{-x}{-1} = x$	M1		For use of $gg(x)$
		A1	4	For correct expression **AG**
(ii)	Order of f = 4	B1		For correct order
	order of g = 2	B1	2	For correct order
(iii)	METHOD 1			
	$y = \dfrac{1}{2-2x} \Rightarrow x = \dfrac{2y-1}{2y}$	M1		For attempt to find inverse
	$\Rightarrow f^{-1}(x) = h(x) = \dfrac{2x-1}{2x}$ *OR* $1 - \dfrac{1}{2x}$	A1	2	For correct expression
	METHOD 2			
	$f^{-1} = f^3 = fg$ or gf	M1		For use of $fg(x)$ or $gf(x)$
	$fg(x) = h(x) = \dfrac{1}{2 - 2\left(\dfrac{1-x}{1-2x}\right)} = \dfrac{1-2x}{-2x}$	A1		For correct expression
(iv)	(table below)	M1		For correct row 1 and column 1
		A1		For e, f, g, h in a latin square
		A1		For correct diagonal e - g - e - g
		A1	4	For correct table
			12	

	e	f	g	h
e	e	f	g	h
f	f	g	h	e
g	g	h	e	f
h	h	e	f	g