ADVANCED GCE

MATHEMATICS 4727

Further Pure Mathematics 3

Candidates answer on the Answer Booklet

OCR Supplied Materials:
- 8 page Answer Booklet
- List of Formulae (MF1)

Other Materials Required:
None

Wednesday 20 May 2009
Afternoon

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

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3R–8H16

1 Find the cube roots of $\frac{1}{2}\sqrt{3} + \frac{1}{2}\mathrm{i}$, giving your answers in the form $\cos\theta + \mathrm{i}\sin\theta$, where $0 \leqslant \theta < 2\pi$. **[4]**

2 It is given that the set of complex numbers of the form $r\mathrm{e}^{\mathrm{i}\theta}$ for $-\pi < \theta \leqslant \pi$ and $r > 0$, under multiplication, forms a group.

(i) Write down the inverse of $5\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}$. **[1]**

(ii) Prove the closure property for the group. **[2]**

(iii) Z denotes the element $\mathrm{e}^{\mathrm{i}\gamma}$, where $\frac{1}{2}\pi < \gamma < \pi$. Express Z^2 in the form $\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta < 0$. **[2]**

3 A line l has equation $\dfrac{x-6}{-4} = \dfrac{y+7}{8} = \dfrac{z+10}{7}$ and a plane p has equation $3x - 4y - 2z = 8$.

(i) Find the point of intersection of l and p. **[3]**

(ii) Find the equation of the plane which contains l and is perpendicular to p, giving your answer in the form $ax + by + cz = d$. **[5]**

4 The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{1-x^2}y = (1-x)^{\frac{1}{2}}, \quad \text{where } |x| < 1,$$

can be solved by the integrating factor method.

(i) Use an appropriate result given in the List of Formulae (MF1) to show that the integrating factor can be written as $\left(\dfrac{1+x}{1-x}\right)^{\frac{1}{2}}$. **[2]**

(ii) Hence find the solution of the differential equation for which $y = 2$ when $x = 0$, giving your answer in the form $y = \mathrm{f}(x)$. **[6]**

5 The variables x and y satisfy the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = \mathrm{e}^{3x}.$$

(i) Find the complementary function. **[3]**

(ii) Explain briefly why there is no particular integral of either of the forms $y = k\mathrm{e}^{3x}$ or $y = kx\mathrm{e}^{3x}$. **[1]**

(iii) Given that there is a particular integral of the form $y = kx^2\mathrm{e}^{3x}$, find the value of k. **[5]**

6 The plane Π_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -5 \\ -2 \end{pmatrix}$.

(i) Express the equation of Π_1 in the form $\mathbf{r.n} = p$. **[4]**

The plane Π_2 has equation $\mathbf{r.} \begin{pmatrix} 7 \\ 17 \\ -3 \end{pmatrix} = 21$.

(ii) Find an equation of the line of intersection of Π_1 and Π_2, giving your answer in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$. **[5]**

7 **(i)** Use de Moivre's theorem to prove that

$$\tan 3\theta \equiv \frac{\tan\theta(3 - \tan^2\theta)}{1 - 3\tan^2\theta}.$$ **[4]**

(ii) **(a)** By putting $\theta = \frac{1}{12}\pi$ in the identity in part **(i)**, show that $\tan\frac{1}{12}\pi$ is a solution of the equation

$$t^3 - 3t^2 - 3t + 1 = 0.$$ **[1]**

(b) Hence show that $\tan\frac{1}{12}\pi = 2 - \sqrt{3}$. **[4]**

(iii) Use the substitution $t = \tan\theta$ to show that

$$\int_0^{2-\sqrt{3}} \frac{t(3 - t^2)}{(1 - 3t^2)(1 + t^2)}\,dt = a \ln b,$$

where a and b are positive constants to be determined. **[5]**

8 A multiplicative group Q of order 8 has elements $\{e, p, p^2, p^3, a, ap, ap^2, ap^3\}$, where e is the identity. The elements have the properties $p^4 = e$ and $a^2 = p^2 = (ap)^2$.

(i) Prove that $a = pap$ and that $p = apa$. **[2]**

(ii) Find the order of each of the elements p^2, a, ap, ap^2. **[5]**

(iii) Prove that $\{e, a, p^2, ap^2\}$ is a subgroup of Q. **[4]**

(iv) Determine whether Q is a commutative group. **[4]**

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4727 Further Pure Mathematics 3

1	$\left(\frac{1}{2}\sqrt{3}+\frac{1}{2}\mathrm{i}\right)^{\frac{1}{3}}=\left(\cos\frac{1}{6}\pi+\mathrm{i}\sin\frac{1}{6}\pi\right)^{\frac{1}{3}}$	B1		For $\arg z=\frac{1}{6}\pi$ seen or implied
	$=\cos\frac{1}{18}\pi+\mathrm{i}\sin\frac{1}{18}\pi$,	M1		For dividing $\arg z$ by 3
	$\cos\frac{13}{18}\pi+\mathrm{i}\sin\frac{13}{18}\pi$,	A1		For any one correct root
	$\cos\frac{25}{18}\pi+\mathrm{i}\sin\frac{25}{18}\pi$	A1	**4**	For 2 other roots and no more in range $0 \leqslant \theta<2\pi$
			4	
2 (i)	$\frac{1}{5}\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}$	B1	**1**	For stating correct inverse in the form $r\,\mathrm{e}^{\mathrm{i}\theta}$
(ii)	$r_1\mathrm{e}^{\mathrm{i}\theta}\times r_2\mathrm{e}^{\mathrm{i}\phi}=r_1r_2\,\mathrm{e}^{\mathrm{i}(\theta+\phi)}$	M1 A1	 **2**	For stating 2 distinct elements multiplied For showing product of correct form
(iii)	$Z^2=\mathrm{e}^{2\mathrm{i}\gamma}$	B1		For $\mathrm{e}^{2\mathrm{i}\gamma}$ seen or implied
	$\Rightarrow\ \mathrm{e}^{2\mathrm{i}\gamma-2\pi\mathrm{i}}$	B1	**2**	For correct answer. aef
			5	
3 (i)	$[6-4\lambda,\,-7+8\lambda,\,-10+7\lambda]$ on p	B1		For point on l seen or implied
	$\Rightarrow 3(6-4\lambda)-4(-7+8\lambda)-2(-10+7\lambda)=8$	M1		For substituting into equation of p
	$\Rightarrow\ \lambda=1\ \Rightarrow\ (2,1,-3)$	A1	**3**	For correct point. Allow position vector
(ii)	METHOD 1			
	$\mathbf{n}=[-4,8,7]\times[3,-4,-2]$	M1* M1 (*dep)		For direction of l and normal of p seen For attempting to find $\mathbf{n}_1\times\mathbf{n}_2$
	$\mathbf{n}=k[12,13,-8]$	A1		For correct vector
	$(2,1,-3)$ OR $(6,-7,-10)$	M1		For finding scalar product of their point on l with their attempt at **n**, or equivalent
	$\Rightarrow\ 12x+13y-8z=61$	A1	**5**	For correct equation, aef cartesian
	METHOD 2			
	$\mathbf{r}=[2,1,-3]$ *OR* $[6,-7,-10]$ $+\lambda[-4,8,7]+\mu[3,-4,-2]$	M1 A1√		For stating eqtn of plane in parametric form (may be implied by next stage), using $[2,1,-3]$ (ft from **(i)**) Or $[6,-7,-10]$, $\mathbf{n}_1$ and $\mathbf{n}_2$ (as above)
	$x=2-4\lambda+3\mu$ $y=1+8\lambda-4\mu$ $z=-3+7\lambda-2\mu$	M1 M1		For writing as 3 linear equations For attempting to eliminate λ and μ
	$\Rightarrow\ 12x+13y-8z=61$	A1		For correct equation aef cartesian
	METHOD 3			
	$3(6+3\mu)-4(-7-4\mu)-2(-10-2\mu)=8$	M1		For finding foot of perpendicular from point on l to p
	$\Rightarrow\ \mu=-2\ \Rightarrow\ (0,1,-6)$	A1		For correct point or position vector
	From 3 points $(2,1,-3)$, $(6,-7,-10)$, $(0,1,-6)$, **n** = vector product of 2 of $[2,0,3]$, $[6,-8,-4]$, $[-4,8,7]$	M1		Use vector product of 2 vectors in plane
	$\Rightarrow\ \mathbf{n}=k[12,13,-8]$			
	$(2,1,-3)$ OR $(6,-7,-10)$	M1		For finding scalar product of their point on l with their attempt at **n**, or equivalent
	$\Rightarrow\ 12x+13y-8z=61$	A1		For correct equation aef cartesian
			8	

4 (i)	IF $e^{\int \frac{1}{1-x^2}\,dx} = e^{\frac{1}{2}\ln\frac{1+x}{1-x}} = \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$	M1 A1	2	For IF stated or implied. Allow $\pm\int$ and omission of dx For integration and simplification to **AG** (intermediate step must be seen)
(ii)	$\frac{d}{dx}\left(y\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}\right) = (1+x)^{\frac{1}{2}}$	M1*		For multiplying both sides by IF
	$y\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} = \frac{2}{3}(1+x)^{\frac{3}{2}} + c$	M1 A1		For integrating RHS to $k(1+x)^n$ For correct equation (including + c)
	$(0, 2) \Rightarrow 2 = \frac{2}{3} + c \Rightarrow c = \frac{4}{3}$	M1 (*dep) M1 (*dep)		In either order: For substituting (0, 2) into their GS (including +c) For dividing solution through by IF, including dividing c or their numerical value for c
	$y = \frac{2}{3}(1+x)(1-x)^{\frac{1}{2}} + \frac{4}{3}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}$	A1	6	For correct solution aef (even unsimplified) in form $y = f(x)$
		8		
5 (i)	$m^2 - 6m + 9\ (= 0) \Rightarrow m = 3$	M1 A1		For attempting to solve correct auxiliary equation For correct m
	CF = $(A + Bx)e^{3x}$	A1	3	For correct CF
(ii)	ke^{3x} and kxe^{3x} both appear in CF	B1	1	For correct statement
(iii)	$y = kx^2e^{3x} \Rightarrow y' = 2kxe^{3x} + 3kx^2e^{3x}$	M1 A1		For differentiating kx^2e^{3x} twice For correct y' aef
	$\Rightarrow y'' = 2ke^{3x} + 12kxe^{3x} + 9kx^2e^{3x}$	A1		For correct y'' aef
	$\Rightarrow$ $ke^{3x}\left(2 + 12x + 9x^2 - 12x - 18x^2 + 9x^2\right) = e^{3x}$	M1		For substituting y'', y', y into DE
	$\Rightarrow k = \frac{1}{2}$	A1	5	For correct k
		9		

6 (i)	METHOD 1			
	$\mathbf{n}_1 = [1, 1, 0] \times [1, -5, -2]$	M1		For attempting to find vector product of the pair of direction vectors
	$= [-2, 2, -6] = k[1, -1, 3]$	A1		For correct $\mathbf{n}_1$
	Use $(2, 2, 1)$	M1		For substituting a point into equation
	$\Rightarrow \mathbf{r}.[-2, 2, -6] = -6 \Rightarrow \mathbf{r}.[1, -1, 3] = 3$	A1	**4**	For correct equation. aef in this form
	METHOD 2			
	$x = 2 + \lambda + \mu$ $y = 2 + \lambda - 5\mu$ $z = 1 \quad - 2\mu$	M1 M1		For writing as 3 linear equations For attempting to eliminate λ and μ
	$\Rightarrow x - y + 3z = 3$	A1		For correct cartesian equation
	$\Rightarrow \mathbf{r}.[1, -1, 3] = 3$	A1		For correct equation. aef in this form
(ii)	For $\mathbf{r} = \mathbf{a} + t\mathbf{b}$			
	METHOD 1			
	$\mathbf{b} = [1, -1, 3] \times [7, 17, -3]$	M1		For attempting to find $\mathbf{n}_1 \times \mathbf{n}_2$
	$= k[2, -1, -1]$	A1√		For a correct vector. ft from $\mathbf{n}_1$ in **(i)**
	e.g. x, y or $z = 0$ in $\begin{cases} x - y + 3z = 3 \\ 7x + 17y - 3z = 21 \end{cases}$	M1		For attempting to find a point on the line
	$\Rightarrow \mathbf{a} = \left[0, \frac{3}{2}, \frac{3}{2}\right]$ OR $[3, 0, 0]$ OR $[1, 1, 1]$	A1√		For a correct vector. ft from equation in **(i)** **SR** a correct vector may be stated without working
	Line is (e.g.) $\mathbf{r} = [1, 1, 1] + t[2, -1, -1]$	A1√	**5**	For stating equation of line ft from **a** and **b** **SR** for $\mathbf{a} = [2, 2, 1]$ stated award M0
	METHOD 2			
	Solve $\begin{cases} x - y + 3z = 3 \\ 7x + 17y - 3z = 21 \end{cases}$ by eliminating one variable (e.g. z)	M1		In either order: For attempting to solve equations
	Use parameter for another variable (e.g. x) to find other variables in terms of t	M1		For attempting to find parametric solution
	(eg) $y = \frac{3}{2} - \frac{1}{2}t$, $z = \frac{3}{2} - \frac{1}{2}t$	A1√ A1√		For correct expression for one variable For correct expression for the other variable ft from equation in **(i)** for both
	Line is (eg) $\mathbf{r} = \left[0, \frac{3}{2}, \frac{3}{2}\right] + t[2, -1, -1]$	A1√		For stating equation of line. ft from parametric solutions
	METHOD 3			
	eg x, y or $z = 0$ in $\begin{cases} x - y + 3z = 3 \\ 7x + 17y - 3z = 21 \end{cases}$	M1		For attempting to find a point on the line
	$\Rightarrow \mathbf{a} = \left[0, \frac{3}{2}, \frac{3}{2}\right]$ *OR* $[3, 0, 0]$ *OR* $[1, 1, 1]$	A1√		For a correct vector. ft from equation in **(i)** **SR** a correct vector may be stated without working **SR** for $\mathbf{a} = [2, 2, 1]$ stated award M0
	eg $[3, 0, 0] - [1, 1, 1]$	M1		For finding another point on the line and using it with the one already found to find **b**
	$\mathbf{b} = k[2, -1, -1]$	A1√		For a correct vector. ft from equation in **(i)**
	Line is (eg) $\mathbf{r} = [1, 1, 1] + t[2, -1, -1]$	A1√		For stating equation of line. ft from **a** and **b**

6 (ii) contd	METHOD 4			
	A point on Π_1 is $[2+\lambda+\mu,\ 2+\lambda-5\mu,\ 1-2\mu]$	M1		For using parametric form for Π_1 and substituting into Π_2
	On $\Pi_2 \Rightarrow$ $[2+\lambda+\mu,\ 2+\lambda-5\mu,\ 1-2\mu].[7, 17, -3]=21$	A1		For correct unsimplified equation
	$\Rightarrow \lambda-3\mu=-1$	A1		For correct equation
	Line is (e.g.) $\mathbf{r}=[2, 2, 1]+(3\mu-1)[1, 1, 0]+\mu[1, -5, -2]$	M1		For substituting into Π_1 for λ *or* μ
	$\Rightarrow \mathbf{r}=[1, 1, 1]$ *or* $\left[\frac{7}{3}, \frac{1}{3}, \frac{1}{3}\right]+t\,[2, -1, -1]$	A1		For stating equation of line
			9	
7 (i)	$\cos 3\theta+\mathrm{i}\sin 3\theta=c^3+3\mathrm{i}c^2s-3cs^2-\mathrm{i}s^3$	M1		For using de Moivre with $n=3$
	$\Rightarrow \cos 3\theta=c^3-3cs^2$ and $\sin 3\theta=3c^2s-s^3$	A1		For both expressions in this form (seen or implied) **SR** For expressions found without de Moivre M0 A0
	$\Rightarrow \tan 3\theta=\dfrac{3c^2s-s^3}{c^3-3cs^2}$	M1		For expressing $\dfrac{\sin 3\theta}{\cos 3\theta}$ in terms of c and s
	$\tan 3\theta=\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}=\dfrac{\tan\theta(3-\tan^2\theta)}{1-3\tan^2\theta}$	A1	**4**	For simplifying to **AG**
(ii) (a)	$\theta=\frac{1}{12}\pi \Rightarrow \tan 3\theta=1$ $\Rightarrow 1-3t^2=t(3-t^2) \Rightarrow$ $t^3-3t^2-3t+1=0$	B1	**1**	For both stages correct **AG**
(b)	$(t+1)(t^2-4t+1)=0$	M1 A1		For attempt to factorise cubic For correct factors
	$\Rightarrow (t=-1),\ t=2\pm\sqrt{3}$	A1		For correct roots of quadratic
	– sign for smaller root $\Rightarrow$ $\tan\frac{1}{12}\pi=2-\sqrt{3}$	A1	**4**	For choice of – sign and correct root **AG**
(iii)	$\mathrm{d}t=(1+t^2)\,\mathrm{d}\theta$	B1		For differentiation of substitution and use of $\sec^2\theta=1+\tan^2\theta$
	$\Rightarrow \int_0^{\frac{1}{12}\pi}\tan 3\theta\,\mathrm{d}\theta$	B1		For integral with correct θ limits seen
	$=\left[\frac{1}{3}\ln(\sec 3\theta)\right]_0^{\frac{1}{12}\pi}=\frac{1}{3}\ln\left(\sec\frac{1}{4}\pi\right)$	M1		For integrating to $k\ln(\sec 3\theta)$ OR $k\ln(\cos 3\theta)$
	$=\frac{1}{3}\ln\sqrt{2}=\frac{1}{6}\ln 2$	M1		For substituting limits and $\sec\frac{1}{4}\pi=\sqrt{2}$ OR $\cos\frac{1}{4}\pi=\frac{1}{\sqrt{2}}$ seen
		A1	**5**	For correct answer aef
			14	

8	(i)	$a^2 = (ap)^2 = apap \Rightarrow a = pap$	B1		For use of given properties to obtain **AG**
		$p^2 = (ap)^2 = apap \Rightarrow p = apa$	B1	2	For use of given properties to obtain **AG** **SR** allow working from **AG** to obtain relevant properties
	(ii)	$(p^2)^2 = p^4 = e \Rightarrow$ order $p^2 = 2$	B1		For correct order with no incorrect working seen
		$(a^2)^2 = (p^2)^2 = e \Rightarrow$ order $a = 4$	B1		For correct order with no incorrect working seen
		$(ap)^4 = a^4 = e \Rightarrow$ order $ap = 4$	B1		For correct order with no incorrect working seen
		$(ap^2)^2 = ap^2ap^2 = ap\,.\,a\,.\,p = a^2$ *OR* $ap^2 = a.a^2 = a^3 \Rightarrow$ $(ap^2)^2 = a^6 = a^2$ $\Rightarrow$ order $ap^2 = 4$	M1 A1	5	For relevant use of **(i)** or given properties For correct order with no incorrect working seen
	(iii)	METHOD 1 $p^2 = a^2,\ ap^2 = a^3$	M2		For use of the given properties to simplify p^2 and ap^2
		$\Rightarrow \{e, a, p^2, ap^2\} = \{e, a, a^2, a^3\}$	A1		For obtaining a^2 and a^3
		which is a cyclic group	A1	4	For justifying that the set is a group

METHOD 2

	e	a	p^2	ap^2
e	e	a	p^2	ap^2
a	a	p^2	ap^2	e
p^2	p^2	ap^2	e	a
ap^2	ap^2	e	a	p^2

M1 For attempting closure with all 9 non-trivial products seen

A1 For all 16 products correct

Completed table is a cyclic group — B2 For justifying that the set is a group

METHOD 3

	e	a	p^2	ap^2
e	e	a	p^2	ap^2
a	a	p^2	ap^2	e
p^2	p^2	ap^2	e	a
ap^2	ap^2	e	a	p^2

M1 For attempting closure with all 9 non-trivial products seen

A1 For all 16 products correct

Identity = e — B1 For stating identity

Inverses exist since
EITHER: e is in each row/column
OR: p^2 is self-inverse; a, ap^2 form an inverse pair — B1 For justifying inverses ($e^{-1} = e$ may be assumed)

(iv)	METHOD 1 e.g. $\left.\begin{array}{l} a\,.\,ap = a^2 p = p^3 \\ ap\,.\,a = p \end{array}\right\} \Rightarrow$ not commutative	M1 M1 B1 A1	 **4**	For attempting to find a non-commutative pair of elements, at least one involving a (may be embedded in a full or partial table) For simplifying elements both ways round For a correct pair of non-commutative elements For stating Q non-commutative, with a clear argument
	METHOD 2 Assume commutativity, so (eg) $ap = pa$	M1		For setting up proof by contradiction
	(i) $\Rightarrow$ $p = ap\,.\,a \Rightarrow p = pa\,.\,a = pa^2 = pp^2 = p^3$	M1		For using **(i)** and/or given properties
	But p and p^3 are distinct	B1		For obtaining and stating a contradiction
	$\Rightarrow Q$ is non-commutative	A1		For stating Q non-commutative, with a clear argument
			15	