Oxford Cambridge and RSA

Monday 22 June 2015 – Morning

A2 GCE MATHEMATICS

4726/01 Further Pure Mathematics 2

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4726/01
- List of Formulae (MF1)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.

- The Question Paper will be found inside the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book**. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.

INFORMATION FOR CANDIDATES

This information is the same on the Printed Answer Book and the Question Paper.

- The number of marks is given in brackets **[ ]** at the end of each question or part question on the Question Paper.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **16** pages. The Question Paper consists of **4** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

- Do not send this Question Paper for marking; it should be retained in the centre or recycled. Please contact OCR Copyright should you wish to re-use this document.

1 By first expressing $\tanh y$ in terms of exponentials, prove that $\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$. **[3]**

2 It is given that $f(x) = \ln(1+\sin x)$. Using standard series, find the Maclaurin series for $f(x)$ up to and including the term in x^3. **[4]**

3 By first completing the square, find the exact value of $\int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{2x-x^2}}\,dx$. **[5]**

4 It is given that $I_n = \int_0^1 x^n e^{-x} dx$ for $n \geqslant 0$.

(i) Show that $I_n = nI_{n-1} + k$ for $n \geqslant 1$, where k is a constant to be determined. **[3]**

(ii) Find the exact value of I_3. **[3]**

(iii) Find the exact value of $990I_8 - I_{11}$. **[3]**

5 It is given that $y = \sin^{-1}2x$.

(i) Using the derivative of $\sin^{-1}x$ given in the List of Formulae (MF1), find $\frac{dy}{dx}$. **[1]**

(ii) Show that $(1-4x^2)\frac{d^2y}{dx^2} = 4x\frac{dy}{dx}$. **[3]**

(iii) Hence show that $(1-4x^2)\frac{d^3y}{dx^3} - 12x\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 0$. **[2]**

(iv) Using your results from parts **(i)**, **(ii)** and **(iii)**, find the Maclaurin series for $\sin^{-1}2x$ up to and including the term in x^3. **[3]**

6 It is given that the equation $3x^3 + 5x^2 - x - 1 = 0$ has three roots, one of which is positive.

(i) Show that the Newton-Raphson iterative formula for finding this root can be written

$$x_{n+1} = \frac{6x_n^{\,3} + 5x_n^{\,2} + 1}{9x_n^{\,2} + 10x_n - 1}.$$ **[3]**

(ii) A sequence of iterates $x_1, x_2, x_3, \ldots$ which will find the positive root is such that the magnitude of the error in x_2 is greater than the magnitude of the error in x_1. On the graph given in the Printed Answer Book, mark a possible position for x_1. **[1]**

(iii) Apply the iterative formula in part **(i)** when the initial value is $x_1 = -1$. Describe the behaviour of the iterative sequence, illustrating your answer on the graph given in the Printed Answer Book. **[2]**

(iv) A sequence of approximations to the positive root is given by $x_1, x_2, x_3, \ldots$. Successive differences

$x_r - x_{r-1} = d_r$, where $r \geqslant 2$, are such that $d_r \approx k(d_{r-1})^2$ where k is a constant.

Show that $d_4 \approx \dfrac{d_3^{\,3}}{d_2^{\,2}}$ and demonstrate this numerically when $x_1 = 1$. **[4]**

(v) Find the value of the positive root correct to 5 decimal places. **[2]**

7 It is given that $\mathrm{f}(x) = \dfrac{x^2 - 25}{(x-1)(x+2)}$.

(i) Express $\mathrm{f}(x)$ in partial fractions. **[4]**

(ii) Write down the equations of the asymptotes of the curve $y = \mathrm{f}(x)$. **[2]**

(iii) Find the value of x where the graph of $y = \mathrm{f}(x)$ cuts the horizontal asymptote. **[2]**

(iv) Sketch the graph of $y^2 = \mathrm{f}(x)$. **[2]**

8 It is given that $\mathrm{f}(x) = 2\sinh x + 3\cosh x$.

(i) Show that the curve $y = \mathrm{f}(x)$ has a stationary point at $x = -\frac{1}{2}\ln 5$ and find the value of y at this point. **[4]**

(ii) Solve the equation $\mathrm{f}(x) = 5$, giving your answers exactly. **[5]**

Question 9 begins on page 4.

9 The equation of a curve in polar coordinates is $r = 2\sin 3\theta$ for $0 \leqslant \theta \leqslant \frac{1}{3}\pi$.

(i) Sketch the curve. **[2]**

(ii) Find the area of the region enclosed by this curve. **[4]**

(iii) By expressing $\sin 3\theta$ in terms of $\sin\theta$, show that a cartesian equation for the curve is

$$(x^2 + y^2)^2 = 6x^2y - 2y^3.$$ **[5]**

END OF QUESTION PAPER

Question			Answer	Marks	Guidance	
1			$\tanh^{-1} x = y \Rightarrow x = \tanh y = \frac{e^{2y}-1}{e^{2y}+1}$	M1	**Oe**	A muddle of x and y unless recovered is M0.
			$(e^{2y}+1)x = e^{2y}-1$			
			$e^{2y}(1-x) = (1+x)$			
			$\Rightarrow e^{2y} = \frac{1+x}{1-x}$	A1	Correct expression for e^{2y} **oe**	
			$2y = \ln\left(\frac{1+x}{1-x}\right)$			
			$(y = \tanh^{-1} x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$	A1	**ag**	
				3		

Question			Answer	Marks	Guidance	
2			$\ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - ...$	B1	Soi. Allow an expansion in x	
			$\sin x = x - \frac{x^3}{6} + ...$	B1	Soi	
			$\ln(1+\sin x) = \left(x - \frac{x^3}{6}\right) - \frac{1}{2}\left(x - \frac{x^3}{6}\right)^2 + \frac{1}{3}\left(x - \frac{x^3}{6}\right)^3 - ...$ $= x - \frac{1}{2}x^2 + x^3\left(\frac{1}{3} - \frac{1}{6}\right)$	M1	For combining series, even if wrong. Must include at least the cubic bracket.	
			$= x - \frac{1}{2}x^2 + \frac{1}{6}x^3$	A1	Ignore further terms www accept 3! for 6	
				4		
			Alternative using Maclaurin general formula $f(x) = \ln(1+\sin x)$ $f(0) = 0$			
			$f'(x) = \frac{\cos x}{(1+\sin x)}$ $f'(0) = 1$	B1	For $f'(x)$	
			$f''(x) = \frac{-1}{(1+\sin x)}$ $f''(0) = -1$	B1	For (not necessarily simplified) $f''(x)$ and $f''(0)$ www	
			$f'''(x) = \frac{\cos x}{(1+\sin x)^2}$ $f'''(0) = 1$			
			Maclaurin: $f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2} + \frac{f'''(0)x^3}{6}$	M1	For correct formula up to 4th term and substituting *their* values	
			$\Rightarrow f(x) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3$	A1	Accept 3! for 6	

Question			Answer	Marks	Guidance	
3			$\int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{2x-x^2}}\,\mathrm{d}x = \int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{1+2x-x^2-1}}\,\mathrm{d}x$	M1	Completing the square on given function	Or
			$= \int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{1-(1-x)^2}}\,\mathrm{d}x$	A1		$= \int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{1-(x-1)^2}}\,\mathrm{d}x$
			$= \left[-\sin^{-1}(1-x)\right]_{\frac{1}{2}}^{1}$	M1	By substitution or using standard form where completed square is of form $1-(1\pm x)^2$	$= \left[\sin^{-1}(x-1)\right]_{\frac{1}{2}}^{1} = \left(0 - -\frac{\pi}{6}\right) = \frac{\pi}{6}$
				A1	Correct result of integration. Ignore limits	
			$= -\left(0 - \frac{\pi}{6}\right) = \frac{\pi}{6}$	A1		
				5		

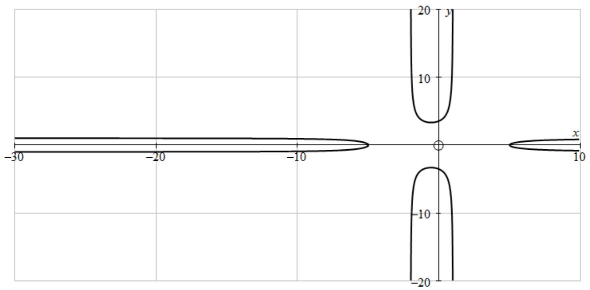
Question			Answer	Marks	Guidance	
4	(i)		$I_n = \int_0^1 x^n \mathrm{e}^{-x}\,\mathrm{d}x \quad u = x^n \quad \mathrm{d}v = \mathrm{e}^{-x}\,\mathrm{d}x$ $\mathrm{d}u = nx^{n-1}\mathrm{d}x \quad v = -\mathrm{e}^{-x}$ $I_n = \left[-\mathrm{e}^{-x}x^n\right]_0^1 + n\int_0^1 x^{n-1}\mathrm{e}^{-x}\,\mathrm{d}x$ $= \left(-\mathrm{e}^{-1} - 0\right) + nI_{n-1}$ $I_n = nI_{n-1} - \mathrm{e}^{-1}$	**M1** **A1** **A1**	By parts Both terms before limits are applied soi Or $k = \frac{-1}{\mathrm{e}}$	
				3		
	(ii)		$I_0 = \int_0^1 \mathrm{e}^{-x}\mathrm{d}x = \left[-\mathrm{e}^{-x}\right]_0^1 = 1 - \mathrm{e}^{-1}$ $I_3 = 3I_2 - \mathrm{e}^{-1}$ $= 3\left(2I_1 - \mathrm{e}^{-1}\right) - \mathrm{e}^{-1} = 6I_1 - 4\mathrm{e}^{-1}$ $= 6(I_0 - \mathrm{e}^{-1}) - 4\mathrm{e}^{-1} = 6I_0 - 10\mathrm{e}^{-1}$ $I_3 = 6 - 16\mathrm{e}^{-1}$	**B1** **M1** **A1**	Complete method even if k is wrong. **SC3** by parts 2 or 3 times	Or finding I_1. Or could be done the other way round.
				3		
	(iii)		$I_{11} = 11I_{10} - \mathrm{e}^{-1}$ $= 11\left(10I_9 - \mathrm{e}^{-1}\right) - \mathrm{e}^{-1} = 110I_9 - 12\mathrm{e}^{-1}$ $= 110(9I_8 - \mathrm{e}^{-1}) - 12\mathrm{e}^{-1} = 990I_8 - 122\mathrm{e}^{-1}$ $990I_8 - I_{11} = 122\mathrm{e}^{-1}$	**M1** **A1** **A1**	Complete method (Could be done the other way round.) For I_{11} in terms of I_{10} or I_9 in terms of I_8 soi	Alternative: Starting from I_4 and working up to I_{11} **M1** I_8 or I_{11} correct **A1** $I_8 = 8! - \frac{109601}{\mathrm{e}}$, $I_{11} = 11! - \frac{108505112}{\mathrm{e}}$
				3		

Question			Answer	Marks	Guidance	
5	(i)		$y=\sin^{-1}(2x)\Rightarrow\frac{dy}{dx}=\frac{1}{\sqrt{1-(2x)^2}}.\frac{d(2x)}{dx}$ $=\frac{2}{\sqrt{1-4x^2}}$	B1	**Oe**	
				1		
	(ii)		$\frac{d^2y}{dx^2}=2\times\left(-\frac{1}{2}\right)(1-4x^2)^{-\frac{3}{2}}(-8x)=\frac{8x}{(1-4x^2)^{\frac{3}{2}}}$ $=\frac{8x}{(1-4x^2)\sqrt{1-4x^2}}=\frac{4x}{(1-4x^2)}\frac{dy}{dx}$ $(1-4x^2)\frac{d^2y}{dx^2}=4x\frac{dy}{dx}$	B1 M1 A1	For correct 2nd derivative Using *their* ans to connect 1st and 2nd derivatives Ft to achieve ag	**SC 2** if result obtained correctly from $y'=\frac{k}{\sqrt{(1-4x^2)}}$
				3		
	(iii)		$(1-4x^2)\frac{d^3y}{dx^3}-8x\frac{d^2y}{dx^2}=4\frac{dy}{dx}+4x\frac{d^2y}{dx^2}$ $(1-4x^2)\frac{d^3y}{dx^3}-12x\frac{d^2y}{dx^2}-4\frac{dy}{dx}=0$	M1 A1	Using result of (ii) and product rule correctly	**M1** Starting with *their* 2nd derivative using appropriate method correctly **A1** ans www
				2		
	(iv)		Find $y_0, y'_0, y''_0, y'''_0=\{0,2,0,8\}$ $y=0+2x+0+\frac{8x^3}{6}$ $\Rightarrow y=2x+\frac{4x^3}{3}$	B1 M1 A1	soi Correctly substituting *their* 4 values into correct Maclaurin www Ignore higher order terms	
				3		

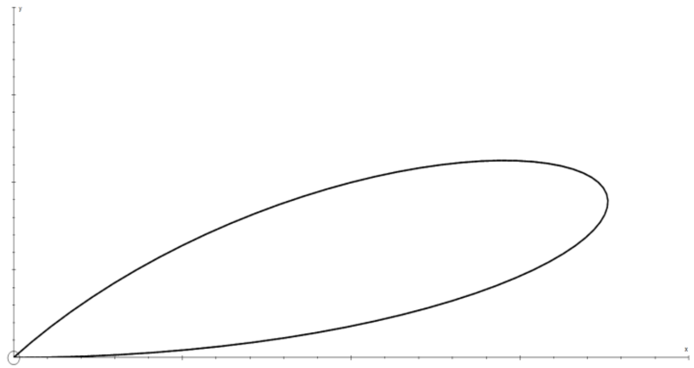
Question			Answer	Marks	Guidance	
6	(i)		$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{3x_n^3+5x_n^2-x_n-1}{9x_n^2+10x_n-1}$	B1	Correct derivative seen	
			$=\frac{x_n(9x_n^2+10x_n-1)-(3x_n^3+5x_n^2-x_n-1)}{9x_n^2+10x_n-1}$	M1	Combining terms seen as 1 fraction or 2 fractions with common denominator	
			$=\frac{9x_n^3+10x_n^2-x_n-3x_n^3-5x_n^2+x_n+1}{9x_n^2+10x_n-1}$			
			$=\frac{6x_n^3+5x_n^2+1}{9x_n^2+10x_n-1}$	A1	Line above seen ag Must contain suffices.	
				3		
	(ii)		A suitable value is shown within range [0.1, 0.25]	B1	The point does not have to be labelled x_1	Accept a tangent which shows this.
				1		
	(iii)		$\Rightarrow x_2=0\Rightarrow x_3=-1$, and statement that values alternate.	B1	Values seen either in words or on graph marked as these values	
			Clear diagram with tangents from – 1 to 0 and back to –1	B1		
				2		
	(iv)		$d_4=kd_3^2,\quad d_3=kd_2^2$	M1	d_4 and d_3 and trying to combine them to eliminate k	
			$\Rightarrow\frac{d_4}{d_3}=\frac{kd_3^2}{kd_2^2}=\frac{d_3^2}{d_2^2}\Rightarrow d_4=\frac{d_3^3}{d_2^2}$	A1	Ag	
			(table below)	B1	Sight of –0.0300	Condone 3 dp
				B1	Sight of –0.0358	3sf or better

x_r	x_{r+1}	d_r		$\frac{d_3^3}{d_2^2}$
1	0.666667	-0.33333	d_2	
0.666667	0.517241	-0.14943	d_3	
0.517241	0.481438	-0.0358	d_4	-0.030
0.481438	0.479363	-0.00207	d_5	
0.479363	0.479356	-6.8E-06		
0.479356	0.479356			

Question			Answer	Marks	Guidance	
				4		
	(v)		Continuing the above to give root 0.47936	**M1** **A1**	Or any other starting point that converges to the positive root Cao	
				2		

Question			Answer	Marks	Guidance	
7	(i)		$\frac{x^2-25}{(x-1)(x+2)} = A + \frac{B}{(x-1)} + \frac{C}{(x+2)}$ $x^2-25 = A(x-1)(x+2)+B(x+2)+C(x-1)$ 3 processes of equating coefficients or substituting: e.g. $x=1 \Rightarrow -24=3B \Rightarrow B=-8$ $x=-2 \Rightarrow -21=-3C \Rightarrow C=7$ coeff of x^2 : $A=1$ $\frac{x^2-25}{(x-1)(x+2)} = 1 - \frac{8}{(x-1)} + \frac{7}{(x+2)}$	M1 B1 A1 A1	Splitting in correct way to give partial fractions (may be seen anywhere) For A For B For C	
				4		
	(ii)		$x = 1, x = -2$ $y = 1$	B1 B1		
				2		
	(iii)		$y=1 \Rightarrow (x-1)(x+2) = x^2-25$ $x^2+x-2 = x^2-25 \Rightarrow x=-23$	M1 A1		
				2		
	(iv)		 	B1 B1	4 bits as shown, roughly symmetric about axes, approaching asymptotes Lh side crosses asymptotes and upper section approaches from above and lower section approaches from below	Ignore any graph of y = f(x)
				2		

Question			Answer	Marks	Guidance	
8	(i)		$y=2\sinh x+3\cosh x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=2\cosh x+3\sinh x$	M1	Diffn and setting = 0	$y=\frac{2}{2}\left(\mathrm{e}^x-\mathrm{e}^{-x}\right)+\frac{3}{2}\left(\mathrm{e}^x+\mathrm{e}^{-x}\right)=\frac{1}{2}\left(5\mathrm{e}^x+\mathrm{e}^{-x}\right)$
			$=0$ when $2\cosh x=-3\sinh x \Rightarrow \tanh x=-\frac{2}{3}$	A1	Correct value for sinh$x$, cosh$x$ or tanh$x$	$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\left(5\mathrm{e}^x-\mathrm{e}^{-x}\right)$
			$x=\tanh^{-1}\left(-\frac{2}{3}\right)=\frac{1}{2}\ln\left(\frac{1-\frac{2}{3}}{1+\frac{2}{3}}\right)=\frac{1}{2}\ln\left(\frac{1}{5}\right)=-\frac{1}{2}\ln 5$	A1	some numerical justification must be seen ag	$=0$ when $5\mathrm{e}^x=\mathrm{e}^{-x} \Rightarrow \mathrm{e}^{2x}=\frac{1}{5} \Rightarrow x=-\frac{1}{2}\ln 5$ Correct exponential form, diffn, set = 0 A1 correct e^{2x} A1 answer
			$\sinh x=\frac{-2}{\sqrt{5}}, \cosh x=\frac{3}{\sqrt{5}} \Rightarrow y=\frac{-4}{\sqrt{5}}+\frac{9}{\sqrt{5}}=\sqrt{5}$	B1	Exact answer only	**SC** Substitute given value of x into derivative to get 0 is 1/3
				4		
	(ii)		$2\sinh x+3\cosh x=5 \Rightarrow 2\frac{\mathrm{e}^x-\mathrm{e}^{-x}}{2}+3\frac{\mathrm{e}^x+\mathrm{e}^{-x}}{2}=5$	M1	Find exponential form	Alt: $\Rightarrow \sqrt{5}\cosh(x+\alpha)=5$ where $\alpha=\frac{1}{2}\ln 5$
			$5\mathrm{e}^x+\mathrm{e}^{-x}=10$ $5\mathrm{e}^{2x}-10\mathrm{e}^x+1=0$	A1	Correct quadratic	$\Rightarrow x=\ln\left(\frac{2+\sqrt{5}}{\sqrt{5}}\right)$ and $-\ln\left(2\sqrt{5}+5\right)$
			$\mathrm{e}^x=\frac{10\pm\sqrt{100-20}}{10}=\frac{10\pm\sqrt{80}}{10}$	M1	Solve *their* 3 term quadratic	
			$x=\ln\left(1+\frac{2\sqrt{5}}{5}\right)$ and $\ln\left(1-\frac{2\sqrt{5}}{5}\right)$	A1 A1	**oe** Single ln only	Penalise only once
				5		

Question			Answer	Marks	Guidance	
9	(i)			B1 B1	Enclosed loop in first quadrant with origin as pole Looking symmetric with line of symmetry around $\theta = \frac{\pi}{6}$ Take one off full marks for more loops	N.B. This means that $\theta = \frac{\pi}{2}$ is not a tangent at the pole.
				2		
	(ii)		Area = $\frac{1}{2}\int_0^{\pi/3} r^2 \mathrm{d}\theta = 2\int_0^{\pi/3} \sin^2 3\theta \,\mathrm{d}\theta$ $= \int_0^{\pi/3} (1-\cos 6\theta)\mathrm{d}\theta = \left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3}$ $= \frac{\pi}{3}$	M1 M1 A1 A1	Correct formula plus limits For obtaining fn in form to integrate using double angle formulae Integral Ft lack of $\frac{1}{2}$ Answer www	Must include $\frac{1}{2}$
				4		
			Alternative: Starting from given equation: Eliminating x and y M1 Get r M1			

Question			Answer	Marks	Guidance	
	(iii)		$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$	M1	Obtaining $\sin 3\theta$ as a function of θ	
			$y = r\sin\theta \Rightarrow \sin\theta = \frac{y}{r}$ and $r^2 = x^2 + y^2$	A1	A correct expression	
			$r = 2\sin 3\theta = 6\sin\theta - 8\sin^3\theta = \frac{6y}{r} - \frac{8y^3}{r^3}$	M1	Eliminate θ	
			$\Rightarrow r^4 = 6yr^2 - 8y^3$	M1	Eliminate r	
			$\Rightarrow \left(x^2 + y^2\right)^2 = 6\left(x^2 + y^2\right)y - 8y^3 = 6x^2y - 2y^3$	A1	ag	
				5		
			Alternative: Starting from given equation: Eliminating x and y **M1** Get r **M1** $r = 6\sin\theta - 8\sin^3\theta$ **A1** Obtain triple angle formula **M1** Ans **A1**			