Monday 23 June 2014 – Morning

A2 GCE MATHEMATICS

4726/01 Further Pure Mathematics 2

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4726/01
- List of Formulae (MF1)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

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INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.

- The Question Paper will be found inside the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book**. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.

INFORMATION FOR CANDIDATES

This information is the same on the Printed Answer Book and the Question Paper.

- The number of marks is given in brackets **[ ]** at the end of each question or part question on the Question Paper.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **16** pages. The Question Paper consists of **4** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

- Do not send this Question Paper for marking; it should be retained in the centre or recycled. Please contact OCR Copyright should you wish to re-use this document.

1 Find $\int_0^2 \frac{1}{\sqrt{4+x^2}}\,dx$, giving your answer exactly in logarithmic form. **[3]**

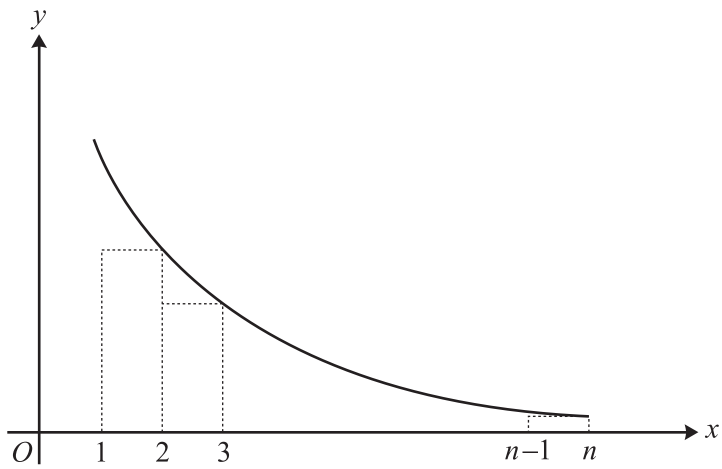
2 It is given that $f(x) = \ln(1+x^2)$.

(i) Using the standard Maclaurin expansion for $\ln(1+x)$, write down the first four terms in the expansion of $f(x)$, stating the set of values of x for which the expansion is valid. **[3]**

(ii) Hence find the exact value of

$$1 - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^4 - \frac{1}{4}\left(\frac{1}{2}\right)^6 + \ldots.$$ **[2]**

3 The diagram shows the curve $y = \frac{1}{x^3}$ for $1 \leqslant x \leqslant n$ where n is an integer. A set of $(n-1)$ rectangles of unit width is drawn under the curve.



(i) Write down the sum of the areas of the rectangles. **[2]**

(ii) Hence show that $\sum_{r=1}^{\infty} \frac{1}{r^3} < \frac{3}{2}$. **[5]**

4 The curves $y = \cos^{-1}x$ and $y = \tan^{-1}(\sqrt{2}x)$ intersect at a point A.

(i) Verify that the coordinates of A are $\left(\frac{1}{\sqrt{2}}, \frac{1}{4}\pi\right)$. **[2]**

(ii) Determine whether the tangents to the curves at A are perpendicular. **[4]**

5 A curve has equation $y = \dfrac{x^2 - 8}{x - 3}$.

(i) Find the equations of the asymptotes of the curve. **[3]**

(ii) Prove that there are no points on the curve for which $4 < y < 8$. **[4]**

(iii) Sketch the curve. Indicate the asymptotes in your sketch. **[2]**

6 **(i)** Given that $y = \cosh^{-1}x$, show that $y = \ln\left(x + \sqrt{x^2 - 1}\right)$. **[4]**

(ii) Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\cosh^{-1}x\right) = \dfrac{1}{\sqrt{x^2 - 1}}$. **[2]**

(iii) Solve the equation $\cosh x = 3$, giving your answers in logarithmic form. **[3]**

7 It is given that, for non-negative integers n, $I_n = \displaystyle\int_0^{\frac{1}{2}\pi} \sin^n x \, \mathrm{d}x$.

(i) Show that $I_n = \dfrac{n-1}{n} I_{n-2}$ for $n \geqslant 2$. **[3]**

(ii) Explain why $I_{2n+1} < I_{2n-1}$. **[2]**

(iii) It is given that $I_{2n+1} < I_{2n} < I_{2n-1}$. Take $n = 5$ to find an interval within which the value of π lies. **[6]**

8 A curve has polar equation $r = a(1 + \cos\theta)$, where a is a positive constant and $0 \leqslant \theta < 2\pi$.

(i) Find the equation of the tangent at the pole. **[2]**

(ii) Sketch the curve. **[2]**

(iii) Find the area enclosed by the curve. **[6]**

9 The equation $10x - 8\ln x = 28$ has a root α in the interval [3, 4]. The iteration $x_{n+1} = \mathrm{g}(x_n)$, where $\mathrm{g}(x) = 2.8 + 0.8\ln x$ and $x_1 = 3.8$, is to be used to find α.

(i) Find the value of α correct to 5 decimal places. You should show the result of each step of the iteration to 6 decimal places. **[4]**

(ii) Illustrate this iteration by means of a sketch. **[2]**

(iii) The difference, δ_r, between successive approximations is given by $\delta_r = x_{r+1} - x_r$. Find δ_3. **[2]**

(iv) Given that $\delta_{n+1} \approx \mathrm{g}'(\alpha)\delta_n$, for all positive integers n, estimate the smallest value of n such that $\delta_n < 10^{-6}\delta_1$. **[4]**

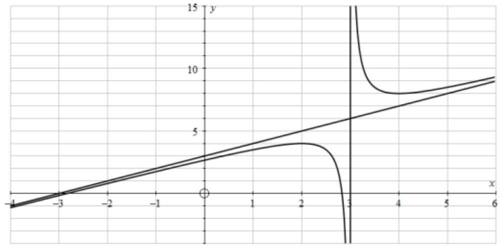
END OF QUESTION PAPER

Question			Answer	Marks	Guidance	
1			$\int_0^2 \frac{1}{\sqrt{4+x^2}}\,dx = \left[\sinh^{-1}\left(\frac{x}{2}\right)\right]_0^2$	M1	Standard form	
			$= \sinh^{-1}1 - \sinh^{-1}0$ $= \ln\left(1+\sqrt{1+1}\right) - 0$	M1	Use of log form and substitute limits dep on 1st M	
			$= \ln\left(1+\sqrt{2}\right)$ **cao isw**	A1		
				[3]		
			Alternative: $\int_0^2 \frac{1}{\sqrt{4+x^2}}\,dx = \left[\ln\left(x+\sqrt{x^2+4}\right)\right]_0^2$	M1	Standard form	
			$= \ln\left(2+\sqrt{8}\right) - \ln 2$	M1	Substitute limits	
			$= \ln\left(1+\sqrt{2}\right)$	A1		

Question			Answer	Marks	Guidance	
2	(i)		$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ $\Rightarrow \ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} \ldots$ Validity: $-1 \le x \le 1$ or $\lvert x \rvert \le 1$	B1 B1 B1 [3]	2 or 3 terms correct unsimplified All terms correct	
	(ii)		$\ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} - \ldots$ Substitute $x = \frac{1}{2}$ $\Rightarrow \ln\left(\frac{5}{4}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{1}{2}\right)^4 + \frac{1}{3}\left(\frac{1}{2}\right)^6 - \frac{1}{4}\left(\frac{1}{2}\right)^8 + \ldots$ $= \frac{1}{4}\left(1 - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^4 - \frac{1}{4}\left(\frac{1}{2}\right)^6 + \ldots\right)$ $\Rightarrow \left(1 - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^4 - \frac{1}{4}\left(\frac{1}{2}\right)^6 + \ldots\right)$ $= 4\ln\left(\frac{5}{4}\right)$ **isw**	M1 A1 [2]	Sub $x = \frac{1}{2}$ into *their* ans to (i) Single ln expression	Alt: divide by x^2 then sub

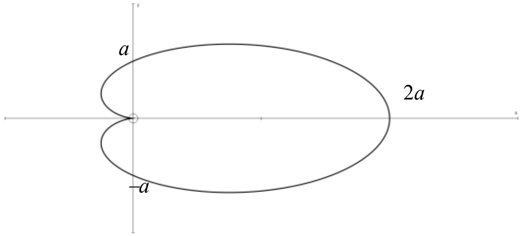
Question			Answer	Marks	Guidance	
3	(i)		Heights of rectangles = $\left(\frac{1}{2}\right)^3, \left(\frac{1}{3}\right)^3, \left(\frac{1}{4}\right)^3, ..., \left(\frac{1}{n}\right)^3$ Width of rectangles = 1 $\Rightarrow$ Sum of areas = $\left(\frac{1}{2}\right)^3 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{4}\right)^3 + ... + \left(\frac{1}{n}\right)^3$ or $\sum_{r=2}^{n}\left(\frac{1}{r}\right)^3$ or $\sum_{r=1}^{n}\left(\frac{1}{r}\right)^3 - 1$	M1 A1 [2]	Heights, with at most one extra and/or one omitted **isw**	 No limits M0
	(ii)		Area = $\int_1^\infty \frac{1}{x^3}\,dx = \left[-\frac{1}{2x^2}\right]_1^\infty = \frac{1}{2}$ Since sum of areas of rectangles approximates, but is less than, the area under the curve $\left(\frac{1}{2}\right)^3 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{4}\right)^3 + ... = \sum_{r=2}^{\infty}\left(\frac{1}{r^3}\right) < \frac{1}{2}$ $\Rightarrow \sum_{r=1}^{\infty}\left(\frac{1}{r^3}\right) < \frac{1}{2} + 1 = \frac{3}{2}$	M1 A1 M1 M1 A1 [5]	Integrate correct function: seen by x^2 in denominator **www** Compare *their* answer to (i) (taken to ∞) with *their* integral dep on 1st M Dealing with 1 Dep on previous 2 Ms	Or with upper limit of *n*

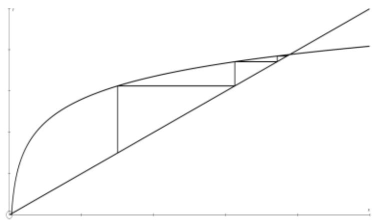
Question		Answer	Marks	Guidance	
4	(i)	For 1st curve $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}$ For 2nd curve $\tan^{-1}\left(\sqrt{2}\times\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}$	B1 B1 [2]		Alt: M1 Set up quadratic in sin or cos and solve A1 Both values correct
	(ii)	For 1st curve $y=\cos^{-1}x$, $\frac{dy}{dx}=\frac{-1}{\sqrt{1-x^2}}$	B1	soi	
		For 2nd curve $y=\tan^{-1}x$, $\frac{dy}{dx}=\frac{\sqrt{2}}{1+2x^2}$	B1	soi	
		For 1st curve, when $x=\frac{1}{\sqrt{2}}$, $\frac{dy}{dx}=\frac{-1}{\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2}}=\frac{-1}{\frac{1}{\sqrt{2}}}=-\sqrt{2}$	M1	Substituting value into *their* derivatives and using $m_1\times m_2=\tilde{}(1)$ (i.e. evidence of finding the product of gradients)	
		For 2nd curve, when $x=\frac{1}{\sqrt{2}}$, $\frac{dy}{dx}=\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}$ Since $m_1\times m_2=-1$ then Yes	A1 [4]	Depends on exact correct numerical values being seen	Acceptable reason: One the negative reciprocal of the other. Condone: One the negative inverse of the other

Question			Answer	Marks	Guidance	
5	(i)		$y=\frac{x^2-8}{x-3}$ Vertical asymptote $x=3$ $y=\frac{x^2-8}{x-3}=\frac{x^2-9+1}{x-3}=\frac{x-3 \quad x+3 \;\; +1}{x-3}$ $=x+3+\frac{1}{x-3}$ $\Rightarrow$ Oblique asymptote: $y=x+3$	B1 M1 A1 [3]	Seen by an answer of $x+a+\left(\frac{b}{x-3}\right)$ Condone incorrect *b*	Allow if fraction missing
	(ii)		$xy-3y=x^2-8 \Rightarrow x^2-xy+\;\; 3y-8 \;\; =0$ Discriminant is $y^2-4 \;\; 3y-8$ $\Rightarrow y^2-12y+32<0 \Rightarrow (y-8)(y-4)<0$ $\Rightarrow 4<y<8$	M1 M1 M1 A1 [4]	Attempt to get quad Finding discriminant Dealing with inequality to find result	Alternative: $\frac{dy}{dx}=1-\frac{1}{x-3^{\;\;2}}$ $=0$ when $x-3^{\;\;2}=1 \Rightarrow x=2,4$ $x=2 \Rightarrow y=4$ $x=4 \Rightarrow y=8$ $\Rightarrow$ No values in range $4<y<8$
	(iii)			B1 B1 [2]	Asymptotes Correct shape	*x* = 3 is identified and the other line has +ve gradient. Must include a vertical and oblique (with +ve gradient) asymptotes and curve must approach them.

Question		Answer	Marks	Guidance	
6	(i)	$x=\cosh y=\frac{e^y+e^{-y}}{2}\Rightarrow e^y+e^{-y}=2x$	M1	Finding 3 term quadratic in e^y	
		$\Rightarrow e^{2y}-2xe^y+1=0$			
		$\Rightarrow e^y=\frac{2x\pm\sqrt{4x^2-4}}{2}=x\pm\sqrt{x^2-1}$	A1	Correct solution	Condone ignoring -ve sign at this point.
		$\Rightarrow y=\ln\ x\pm\sqrt{x^2-1}$			Condone interchange of *x* and *y* but final ans must be correct
		Reject $-$ sign as principal value taken	B1	Including reason **oe**	
		$\Rightarrow y=\ln\ x+\sqrt{x^2-1}$	A1		
			[4]		
	(ii)	$y=\ln\ x+\sqrt{x^2-1}\ \Rightarrow\frac{dy}{dx}=\frac{1}{x+\sqrt{x^2-1}}\times\left(1+\frac{2x}{2\sqrt{x^2-1}}\right)$	M1	Alt: $x=\cosh y\Rightarrow\frac{dy}{dx}=\frac{1}{\sinh y}$	
		$=\frac{1}{x+\sqrt{x^2-1}}\times\frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}=\frac{1}{\sqrt{x^2-1}}$	A1	$=\frac{1}{\sqrt{x^2-1}}$	
			[2]		
	(iii)	$x=\cosh^{-1}3$	M1	Use of $\cosh^{-1}$	
		$=\ln\ 3+\sqrt{8}$	A1		
		$=-\ln\ 3+\sqrt{8}$ **oe**	A1	**ft**, -ve the first answer	
			[3]		

Question			Answer	Marks	Guidance	
7	(i)		$I_n = \int_0^{\pi/2} \sin^n x \, dx = I_n = \int_0^{\pi/2} \sin^{n-1} x \sin x \, dx$	M1	Correct start for reduction	
			$\Rightarrow I_n = \left[\sin^{n-1} x \times -\cos x\right]_0^{\pi/2} + \int_0^{\pi/2} \cos x (n-1) \sin^{n-2} x \cos x \, dx$			
			$= 0 + (n-1) \int_0^{\pi/2} \sin^{n-2} x \; 1 - \sin^2 x \; dx = (n-1) \; I_{n-2} - I_n$	M1	Deal with $\cos^2$ dep on 1st M	
			$\Rightarrow nI_n = I_{n-2} \Rightarrow I_n = \frac{n-1}{n} I_{n-2}$	A1	**www**	
				[3]		
	(ii)		$I_n = \frac{n-1}{n} I_{n-2} \Rightarrow I_{2n+1} = \frac{2n+1 \; -1}{2n+1} I_{2n-1} = \frac{2n}{2n+1} I_{2n-1}$	M1	Allow using *n* instead of 2*n*+ 1	Alt: M1 $y = \sin^n x < y = \sin^{n-2} x$ in range
			and $\frac{2n}{2n+1} < 1$	A1		A1 means that the area underneath is less and therefore...
			Alternative $I_n = \frac{n-1}{n} I_{n-2} . \frac{n-1}{n} < 1 \Rightarrow I_n < I_{n-2}$ for all $n \Rightarrow I_{2n+1} < I_{2n-1}$			This can be argued one step at a time instead of 2
				[2]		
	(iii)		$I_{11} = \frac{256}{693}, \quad I_{10} = \frac{63}{512}.\pi, \quad I_9 = \frac{128}{315}$	B1 B1	For I_1 soi For I_0 soi	Allow for pa.
			$\Rightarrow \frac{256}{693} < \frac{63}{512}.\pi < \frac{128}{315}$	M1	Applying reduction formula for at least one of I_9 and I_{11}	
			$\Rightarrow \frac{131072}{43659} < \pi < \frac{65536}{19845}$	M1	Applying reduction formula for I_{10}	
			$\Rightarrow 3.0022 < \pi < 3.3024$	A1	Lhs fraction or decimal equivalent correct to 4dp	Both correct but both only to 3sf give A1 only
				A1	Likewise Rhs	
				[6]		

Question			Answer	Marks	Guidance	
8	(i)		$a\ 1+\cos\theta\ =0$ when $\cos\theta=-1$ $\Rightarrow\theta=\pi$	M1 A1 [2]	**soi** Only this answer: A0 if anything else	
	(ii)			B1 B1 [2]	Correct shape, correct orientation, roughly symmetric All 3 intersections on axes indicated, cusp at pole dep on 1st B.	
	(iii)		$r=a(1+\cos\theta)$ $A=\frac{1}{2}\int_0^{2\pi}r^2\,d\theta=\frac{1}{2}\int_0^{2\pi}a^2(1+\cos\theta)^2\,d\theta$ $=\frac{a^2}{2}\int_0^{2\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$ $=\frac{a^2}{2}\int_0^{2\pi}(1+2\cos\theta+\frac{1}{2}\ \cos2\theta+1\ )\,d\theta$ $=\frac{a^2}{2}\left[\theta+2\sin\theta+\frac{1}{2}\left(\frac{1}{2}\sin2\theta+\theta\right)\right]_0^{2\pi}=\frac{a^2}{2}\left(2\pi+0+\frac{1}{2}\ 0+2\pi\ \right)$ $=\frac{a^2}{2}\ 3\pi\ =\frac{3\pi a^2}{2}$	M1 A1 M1 A1 M1 A1 [6]	Use of formula with limits Condone omission of a^2 Dealing with $\cos^2$ Condone omission of a^2 Substitute limits dep on 2nd M	

Question			Answer	Marks	Guidance	
9	(i)		3.8 3.868001 3.868001 3.882190 3.882190 3.885120 3.885120 3.885723 3.885723 3.885847 3.885847 3.885873 3.885873 3.885878 Root = 3.88588	M1 A1 A1 A1 [4]	 For x_2 For x_3	N.B. Working must be seen
	(ii)			B1 B1 [2]	Curve and line Iterations showing staircase from below. At least two seen	Concave curve initially above $y = x$ Only [3,4] required so ignore behaviour at origin
	(iii)		3.8 3.868001 0.068001 □$_1$ 3.868001 3.88219 0.014189 □$_2$ 3.88219 3.88512 0.002929 □$_3$ 3.88512 3.885723 0.000603 3.885723 3.885847 0.000124 3.885847 3.885873 3.885873 3.885878 □$_3$ = 0.00293	M1 A1 [2]	Working differences Anything that rounds to 0.00293	

Question			Answer	Marks	Guidance	
	(iv)		$g'(\alpha) = \frac{0.8}{3.88588} = 0.20587$ $g'(\alpha)^{n-1} < 10^{-6}$ $\Rightarrow\ n-1\ \log 0.20587 < \log 10^{-6}$ $\Rightarrow n-1 > \frac{6}{.68640} = 8.74...$ $\Rightarrow n > 9.74....$ i.e. least $n = 10$	M1 A1 M1 A1 [4]	Attempt to find g' by differentiating g(x) correctly. Condone = Take logs If = has been used then the answer must include a justification	S.C. by successive evaluations B4 S.C. answer only seen B2 If ans wrong: M1 for g', M1 for successive multiplication by g' Or: M1 for continuation of table to find d4, d5, etc and a comparison with 10^{-6}d1