

Thursday 13 June 2013 – Morning

A2 GCE MATHEMATICS

4726/01 Further Pure Mathematics 2

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:

- Printed Answer Book 4726/01
- List of Formulae (MF1)

Other materials required:

- Scientific or graphical calculator

Duration: 1 hour 30 minutes



INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.

- The Question Paper will be found in the centre of the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book.** Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.

INFORMATION FOR CANDIDATES

This information is the same on the Printed Answer Book and the Question Paper.

- The number of marks is given in brackets [] at the end of each question or part question on the Question Paper.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **12** pages. The Question Paper consists of **4** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER/INVIGILATOR

- Do not send this Question Paper for marking; it should be retained in the centre or recycled. Please contact OCR Copyright should you wish to re-use this document.

1 By using the substitution $t = \tan \frac{1}{2}\theta$, find $\int_0^{\frac{1}{2}\pi} \frac{1}{1 + \cos \theta} d\theta$. [5]

2 (i) Using the definitions for $\cosh x$ and $\sinh x$ in terms of e^x and e^{-x} , show that $\cosh^2 x - \sinh^2 x \equiv 1$. [3]

(ii) Hence solve the equation $\sinh^2 x = 5 \cosh x - 7$, giving your answers in logarithmic form. [5]

3 It is given that $f(x) = \tanh^{-1}\left(\frac{1-x}{3+x}\right)$ for $x > -1$.

(i) Show that $f''(x) = \frac{1}{2(x+1)^2}$. [6]

(ii) Hence find the Maclaurin series for $f(x)$ up to and including the term in x^2 . [4]

4 It is given that $I_n = \int_0^{\frac{1}{2}\pi} \cos^n x dx$ for $n \geq 0$.

(i) Show that $I_n = \frac{n-1}{n} I_{n-2}$ for $n \geq 2$. [5]

(ii) Hence find I_{11} as an exact fraction. [3]

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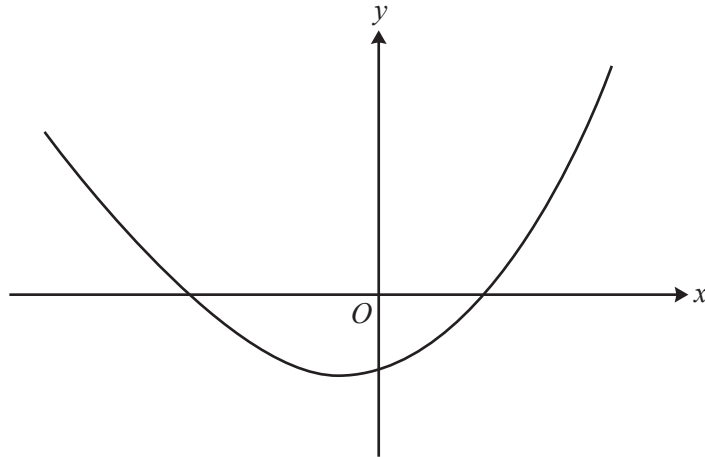
5 You are given that the equation $x^3 + 4x^2 + x - 1 = 0$ has a root, α , where $-1 < \alpha < 0$.

(i) Show that the Newton-Raphson iterative formula for this equation can be written in the form

$$x_{n+1} = \frac{2x_n^3 + 4x_n^2 + 1}{3x_n^2 + 8x_n + 1}. \quad [3]$$

(ii) Using the initial value $x_1 = -0.7$, find x_2 and x_3 and find α correct to 5 decimal places. [3]

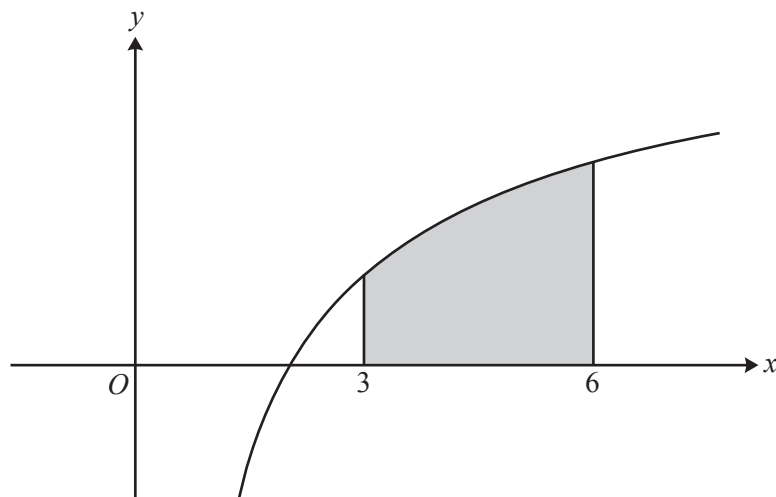
(iii) The diagram shows a sketch of the curve $y = x^3 + 4x^2 + x - 1$ for $-1.5 \leq x \leq 1$.



Using the copy of the diagram in your answer book, explain why the initial value $x_1 = 0$ will fail to find α . [2]

[Questions 6, 7 and 8 are printed overleaf.]

6



The diagram shows part of the curve $y = \ln(\ln(x))$. The region between the curve and the x -axis for $3 \leq x \leq 6$ is shaded.

(i) By considering n rectangles of equal width, show that a lower bound, L , for the area of the shaded region is $\frac{3}{n} \sum_{r=0}^{n-1} \ln\left(\ln\left(3 + \frac{3r}{n}\right)\right)$. [3]

(ii) By considering another set of n rectangles of equal width, find a similar expression for an upper bound, U , for the area of the shaded region. [1]

(iii) Find the least value of n for which $U - L < 0.001$. [4]

7 The equation of a curve is $y = \frac{x^2 + 1}{(x + 1)(x - 7)}$.

(i) Write down the equations of the asymptotes. [3]

(ii) Find the coordinates of the stationary points on the curve. [5]

(iii) Find the coordinates of the point where the curve meets one of its asymptotes. [3]

(iv) Sketch the curve. [3]

8 The equation of a curve is $x^2 + y^2 - x = \sqrt{x^2 + y^2}$.

(i) Find the polar equation of this curve in the form $r = f(\theta)$. [3]

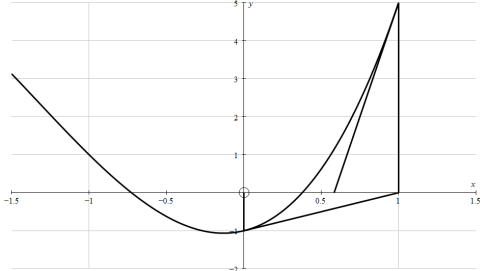
(ii) Sketch the curve. [2]

(iii) The line $x + 2y = 2$ divides the region enclosed by the curve into two parts. Find the ratio of the two areas. [6]

Question	Answer	Marks	Guidance
1	$\cos \theta = \frac{1-t^2}{1+t^2}$ $\frac{dt}{d\theta} = \frac{1}{2} \sec^2 \frac{1}{2} \theta = \frac{1}{2} \left(1 + \tan^2 \frac{1}{2} \theta \right)$ $\Rightarrow dt = \frac{1+t^2}{2} \cdot d\theta \Rightarrow d\theta = \frac{2dt}{1+t^2}$ $\Rightarrow I = \int_0^1 \frac{1}{1 + \frac{1-t^2}{1+t^2}} \frac{2dt}{1+t^2} = \int_0^1 \frac{1+t^2}{1+t^2+1-t^2} \frac{2dt}{1+t^2}$ $\int_0^1 \frac{2dt}{2} = [t]_0^1 = 1$	M1 A1 M1 A1 A1 [5]	Using t substitution for both $\cos \theta$ and $d\theta$ Subs correct Dealing with limits and attempting integration. Correct integral Answer
	Alternative $1 + \cos \theta = 2 \cos^2 \frac{1}{2} \theta$ $\Rightarrow \int_0^{\frac{\pi}{2}} \frac{1}{1 + \cos \theta} d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{1}{\cos^2 \frac{1}{2} \theta} d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} \sec^2 \frac{1}{2} \theta d\theta$ $= \frac{1}{2} \left[2 \tan \frac{1}{2} \theta \right]_0^{\frac{\pi}{2}} = \tan \frac{\pi}{2} - \tan 0 = 1$	SC3	
2 (i)	$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}$ $\Rightarrow \cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2} \right)^2 - \left(\frac{e^x - e^{-x}}{2} \right)^2$ $= \frac{1}{4} (e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}) = \frac{1}{4} \cdot 4 = 1$	B1 M1 A1 [3]	Correct formulae Dealing with squaring correctly www All steps seen Difference of squares can be used

Question	Answer	Marks	Guidance
2 (ii)	$\Rightarrow \cosh^2 x - 1 = 5 \cosh x - 7$ $\Rightarrow \cosh^2 x - 5 \cosh x + 6 = 0$ $\Rightarrow (\cosh x - 2)(\cosh x - 3) = 0$ $\Rightarrow \cosh x = 2, 3$ $\Rightarrow x = \cosh^{-1} 2 = \pm \ln(2 \pm \sqrt{3})$ <p style="text-align: center;">and $x = \cosh^{-1} 3 = \pm \ln(3 \pm \sqrt{8})$</p>	M1 M1 A1 A1 A1 [5]	Use (i) Attempt to solve quadratic Use correct ln formula Use correct ln formula E.g. correct formula or expansion of their brackets gives 2 out of 3 terms correct Condone lack of \pm Condone lack of \pm
3 (i)	$\frac{dy}{dx} = \frac{1}{1 - \left(\frac{1-x}{3+x}\right)^2} \times \frac{-(3+x) - (1-x)}{(3+x)^2}$ $\Rightarrow \frac{dy}{dx} = \left(\frac{-4}{(3+x)^2 - (1-x)^2} \right) = \frac{k}{1+x}$ $\Rightarrow \frac{dy}{dx} = \frac{-1}{2(1+x)}$ $\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2(1+x)^2}$	B1 M1 A1 A1 A1 A1 [6]	Sight of standard diffn for $\tanh^{-1}x$ Fn of fn and diffn of quotient Soi correct quotient (i.e. correct expression for 2nd part) Correct for y' 2 nd diffn (NB AG)

Question	Answer	Marks	Guidance
3 (ii)	When $x = 0, y = \tanh^{-1} \frac{1}{3}$ or $\frac{1}{2} \ln 2$ or $\ln \sqrt{2}$ $\frac{dy}{dx} = -\frac{1}{2}$ $\frac{d^2y}{dx^2} = \frac{1}{2}$ $\Rightarrow y = \tanh^{-1} \frac{1}{3} + \left(-\frac{1}{2}\right)x + \left(\frac{1}{2}\right)\frac{x^2}{2}$ $= \tanh^{-1} \frac{1}{3} - \frac{1}{2}x + \frac{x^2}{4}$	B1 B1 M1 A1 [4]	For 1 st value (needs to be exact) For both Use of correct Maclaurin's series Accept 0.347
4 (i)	$u = \cos^{n-1} x, dv = \cos x dx$ $du = -(n-1)\cos^{n-2} x \sin x, v = \sin x$ $\Rightarrow I_n = \left[\cos^{n-1} x \sin x\right]_0^{\frac{\pi}{2}} + (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x dx$ $= 0 + (n-1)(I_{n-2} - I_n)$ $\Rightarrow nI_n = (n-1)I_{n-2} \Rightarrow I_n = \frac{n-1}{n} I_{n-2}$	M1* A1 A1 *M1 A1 [5]	By parts the right way round Integral so far Correct use of $\sin^2 x = 1 - \cos^2 x$ Dependent on 1st M www AG
4 (ii)	$I_1 = 1$ $I_{11} = \frac{10}{11} I_9 = \dots = \frac{10}{11} \cdot \frac{8}{9} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} I_1$ $\Rightarrow I_{11} = \frac{3840}{10395} = \frac{256}{693}$ oe	B1 M1 A1 [3]	For I_1 soi Use of (i) to give product of 5 fractions Correct fraction

Question	Answer	Marks	Guidance	
5 (i)	$f(x) = x^3 + 4x^2 + x - 1$ $f'(x) = 3x^2 + 8x + 1$ $\Rightarrow x_{n+1} = x_n - \frac{x_n^3 + 4x_n^2 + x_n - 1}{3x_n^2 + 8x_n + 1}$ $= \frac{x_n(3x_n^2 + 8x_n + 1) - (x_n^3 + 4x_n^2 + x_n - 1)}{3x_n^2 + 8x_n + 1}$ $= \frac{2x_n^3 + 4x_n^2 + 1}{3x_n^2 + 8x_n + 1}$	B1 M1 A1 [3]	Diffn Correct application of N-R formula And completed with suffices on last line NB AG	
5 (ii)	$x_2 = -0.72652,$ $x_3 = -0.72611$ $\Rightarrow \alpha = -0.72611$	B1 B1 B1 [3]		NB $x_4 = -0.726109$
5 (iii)	Sketch plus at least one tangent  Converges to another root.	B1 B1 [2]	At least the first tangent and vertical line to curve Or positive root or, for e.g. "x = 0 is the wrong side of a turning point" www	Use of formula to find this root numerically is not acceptable

Question		Answer	Marks	Guidance
6	(i)	Width of rectangles is $\frac{3}{n}$ \Rightarrow Sum of areas of rectangles $= \frac{3}{n} \times \left(\ln(\ln 3) + \ln \left(\ln \left(3 + \frac{3}{n} \right) \right) + \dots \right)$ $= \frac{3}{n} \times \sum_{r=0}^{n-1} \ln \left(\ln \left(3 + \frac{3r}{n} \right) \right)$	B1 M1 A1 [3]	Statement about width Height or area of at least one rectangle Correct conclusion www 1468 or
6	(ii)	$= \frac{3}{n} \times \sum_{r=1}^n \ln \left(\ln \left(3 + \frac{3r}{n} \right) \right)$	B1 [1]	
6	(iii)	$U - L = \frac{3}{n} \times \ln(\ln 6) - \frac{3}{n} \times \ln(\ln 3)$ $= \frac{3}{n} (\ln(\ln 6) - \ln(\ln 3)) = \frac{3}{n} \ln \left(\frac{\ln 6}{\ln 3} \right)$ $\Rightarrow n > \frac{3}{0.001} \ln \left(\frac{\ln 6}{\ln 3} \right) \Rightarrow n > \frac{3}{0.001} \times \ln(1.6309)$ $\Rightarrow \text{least } n = 1468$	M1* A1 *M1 A1 [4]	Subtraction to obtain the difference of two terms Dealing with inequality to obtain n dep on first M Accept $n \geq 1468$ or $n > 1467$
7	(i)	$x = -1$ $x = 7$ $y = 1$	B1 B1 B1 [3]	B1 for each -1 for any extras

Question	Answer	Marks	Guidance
7 (ii)	$\frac{dy}{dx} = \frac{(x^2 - 6x - 7)2x - (x^2 + 1)(2x - 6)}{(x+1)^2(x-7)^2}$ $= 0 \text{ when } (x^2 - 6x - 7)2x - (x^2 + 1)(2x - 6) = 0$ $3x^2 + 8x - 3 = 0$ $\Rightarrow x = -3, \frac{1}{3}; \quad y = \frac{1}{2}, -\frac{1}{8}$ <p>i.e. $\left(-3, \frac{1}{2}\right), \left(\frac{1}{3}, -\frac{1}{8}\right)$</p>	M1 A1 A1 A1 A1 [5]	Diffn using quotient rule Quadratic Both x values Both y values Or: A1 one pair A1 other pair
7 (iii)	When $y = 1$, $x^2 - 6x - 7 = x^2 + 1$ $\Rightarrow 6x = -8 \Rightarrow x = -\frac{4}{3} \Rightarrow \left(-\frac{4}{3}, 1\right)$	M1 A1 A1 [3]	Coordinate pair needs to be seen.
7 (iv)		B1 B1 B1 [3]	Left section, cutting asymptote and approaching $y = 1$ from below Right hand section Middle section all below x -axis labelling intercept on graph or by a statement

