OCR

RECOGNISING ACHIEVEMENT

**ADVANCED GCE**

# MATHEMATICS **4726**

Further Pure Mathematics 2

Candidates answer on the Answer Booklet

**OCR Supplied Materials:**
- 8 page Answer Booklet
- List of Formulae (MF1)

**Other Materials Required:**
None

**Monday 11 January 2010**
**Morning**

**Duration:** 1 hour 30 minutes

*4726*

**INSTRUCTIONS TO CANDIDATES**

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a graphical calculator in this paper.

**INFORMATION FOR CANDIDATES**

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

**1** It is given that $f(x) = x^2 - \sin x$.

**(i)** The iteration $x_{n+1} = \sqrt{\sin x_n}$, with $x_1 = 0.875$, is to be used to find a real root, $\alpha$, of the equation $f(x) = 0$. Find $x_2$, $x_3$ and $x_4$, giving the answers correct to 6 decimal places. **[2]**

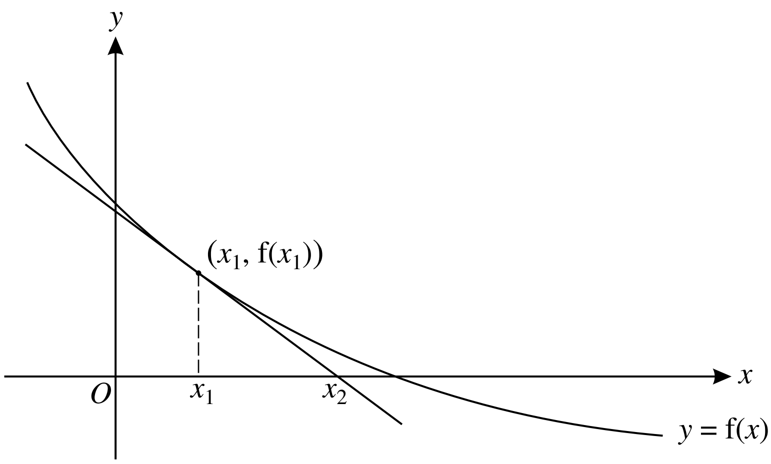
**(ii)** The error $e_n$ is defined by $e_n = \alpha - x_n$. Given that $\alpha = 0.876\,726$, correct to 6 decimal places, find $e_3$ and $e_4$. Given that $g(x) = \sqrt{\sin x}$, use $e_3$ and $e_4$ to estimate $g'(\alpha)$. **[3]**

**2** It is given that $f(x) = \tan^{-1}(1 + x)$.

**(i)** Find $f(0)$ and $f'(0)$, and show that $f''(0) = -\frac{1}{2}$. **[4]**

**(ii)** Hence find the Maclaurin series for $f(x)$ up to and including the term in $x^2$. **[2]**

**3**




A curve with no stationary points has equation $y = f(x)$. The equation $f(x) = 0$ has one real root $\alpha$, and the Newton-Raphson method is to be used to find $\alpha$. The tangent to the curve at the point $(x_1, f(x_1))$ meets the $x$-axis where $x = x_2$ (see diagram).

**(i)** Show that $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$. **[3]**

**(ii)** Describe briefly, with the help of a sketch, how the Newton-Raphson method, using an initial approximation $x = x_1$, gives a sequence of approximations approaching $\alpha$. **[2]**

**(iii)** Use the Newton-Raphson method, with a first approximation of 1, to find a second approximation to the root of $x^2 - 2\sinh x + 2 = 0$. **[2]**

**4** The equation of a curve, in polar coordinates, is

$$r = e^{-2\theta}, \quad \text{for } 0 \leqslant \theta \leqslant \pi.$$

**(i)** Sketch the curve, stating the polar coordinates of the point at which $r$ takes its greatest value. **[2]**

**(ii)** The pole is $O$ and points $P$ and $Q$, with polar coordinates $(r_1, \theta_1)$ and $(r_2, \theta_2)$ respectively, lie on the curve. Given that $\theta_2 > \theta_1$, show that the area of the region enclosed by the curve and the lines $OP$ and $OQ$ can be expressed as $k(r_1^2 - r_2^2)$, where $k$ is a constant to be found. **[5]**

**5** **(i)** Using the definitions of $\sinh x$ and $\cosh x$ in terms of $e^x$ and $e^{-x}$, show that

$$\cosh^2 x - \sinh^2 x \equiv 1.$$

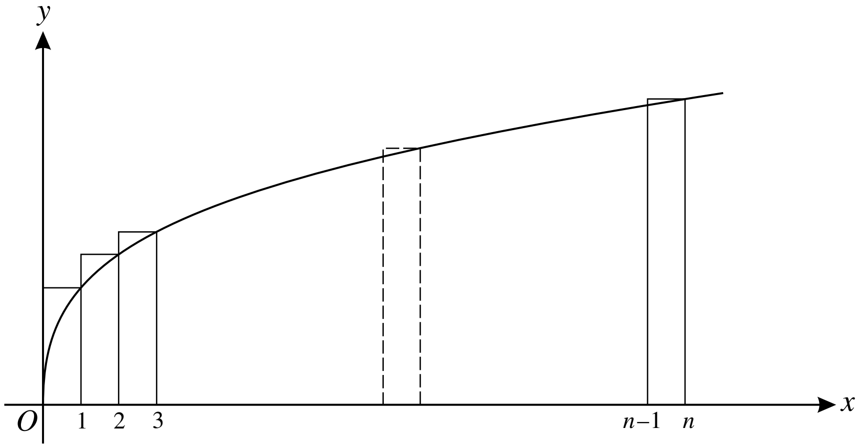
Deduce that $1 - \tanh^2 x \equiv \operatorname{sech}^2 x$. **[4]**

**(ii)** Solve the equation $2\tanh^2 x - \operatorname{sech} x = 1$, giving your answer(s) in logarithmic form. **[4]**

**6** **(i)** Express $\dfrac{4}{(1-x)(1+x)(1+x^2)}$ in partial fractions. **[5]**

**(ii)** Show that $\displaystyle\int_0^{\frac{1}{\sqrt{3}}} \frac{4}{1-x^4}\,dx = \ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) + \tfrac{1}{3}\pi$. **[4]**

**7**




The diagram shows the curve with equation $y = \sqrt[3]{x}$, together with a set of $n$ rectangles of unit width.

**(i)** By considering the areas of these rectangles, explain why

$$\sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \dots + \sqrt[3]{n} > \int_0^n \sqrt[3]{x}\,dx.$$ **[2]**

**(ii)** By drawing another set of rectangles and considering their areas, show that

$$\sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \dots + \sqrt[3]{n} < \int_1^{n+1} \sqrt[3]{x}\,dx.$$ **[3]**

**(iii)** Hence find an approximation to $\displaystyle\sum_{n=1}^{100} \sqrt[3]{n}$, giving your answer correct to 2 significant figures. **[3]**

**[Questions 8 and 9 are printed overleaf.]**

**8** The equation of a curve is

$$y = \frac{kx}{(x-1)^2},$$

where $k$ is a positive constant.

**(i)** Write down the equations of the asymptotes of the curve. **[2]**

**(ii)** Show that $y \geqslant -\frac{1}{4}k$. **[4]**

**(iii)** Show that the $x$-coordinate of the stationary point of the curve is independent of $k$, and sketch the curve. **[4]**

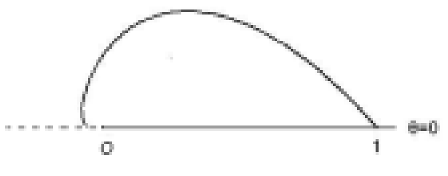
**9** **(i)** Given that $y = \tanh^{-1} x$, for $-1 < x < 1$, prove that $y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$. **[3]**

**(ii)** It is given that $f(x) = a\cosh x - b\sinh x$, where $a$ and $b$ are positive constants.

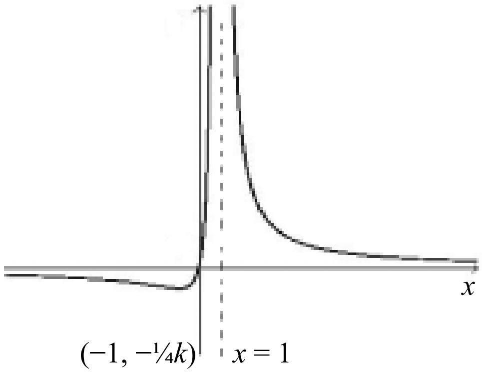
**(a)** Given that $b \geqslant a$, show that the curve with equation $y = f(x)$ has no stationary points. **[3]**

**(b)** In the case where $a > 1$ and $b = 1$, show that $f(x)$ has a minimum value of $\sqrt{a^2 - 1}$. **[6]**

# 4726 Further Pure Mathematics 2

| | | | | |
|---|---|---|---|---|
| **1** | **(i)** | Get 0.876096, 0.876496, 0.876642 | B1√<br>B1 | For any one correct or √ from wrong answer; radians only<br>All correct |
| | **(ii)** | Subtract correctly (0.00023(0), 0.000084)<br>Divide their errors as $e_4/e_3$ only<br>Get 0.365(21…) | B1√<br>M1<br>A1 | On their answers<br>May be implied<br>Cao |
| **2** | **(i)** | Find $f'(x) = 1/(1+(1+x)^2)$<br><br>Get $f(0) = \frac{1}{4}\pi$ and $f'(0) = \frac{1}{2}$<br>Attempt $f''(x)$<br><br>Correctly get $f''(0) = -\frac{1}{2}$ | M1<br><br>A1√<br>M1<br><br>A1 | Quoted or derived; may be simplified or left as $\sec^2 y \, dy/dx = 1$<br>On their $f'(0)$; allow $f(0)=0.785$ but not 45<br>Reasonable attempt at chain/quotient rule or implicit differentiation<br>A.G. |
| | **(ii)** | Attempt Maclaurin as $af(0)+bf'(0)+cf''(0)$<br>Get $\frac{1}{4}\pi + \frac{1}{2}x - \frac{1}{4}x^2$ | M1<br>A1 | Using their f(0) and f′(0)<br>Cao; allow 0.785 |
| **3** | **(i)** | Attempt gradient as $\pm f(x_1)/(x_2 - x_1)$<br>Equate to gradient of curve at $x_1$<br>Clearly arrive at A.G.<br><br>**SC** Attempt equation of tangent<br>Put $(x_2, 0)$ into their equation<br>Clearly arrive at A.G. | M1<br>M1<br>A1<br><br>M1<br>M1<br>A1 | Allow reasonable $y$-step/$x$-step<br>Allow ±<br>Beware confusing use of ±<br><br>As $y - f(x_1) = f'(x_1)(x - x_1)$ |
| | **(ii)** | Diagram showing at least one more tangent<br>Description of tangent meeting $x$-axis, used as next starting value | B1<br><br>B1 | |
| | **(iii)** | Reasonable attempt at N-R<br>Get 1.60 | M1<br>A1 | Clear attempt at differentiation<br>Or answer which rounds |
| **4** | **(i)** | State $r = 1$ and $\theta = 0$.<br> | B1<br><br>B1 | May be seen or implied<br><br>Correct shape, decreasing $r$ (not through $O$) |
| | **(ii)** | Use $\frac{1}{2}\int r^2 \, d\theta$ with $r = e^{-2\theta}$ seen or implied<br>Integrate correctly as $-\frac{1}{8}e^{-4\theta}$<br>Use limits in correct order<br>Use $r_1^2 = e^{-4\theta}$ etc.<br>Clearly get $k = \frac{1}{8}$ | M1<br>A1<br>M1<br>M1<br>A1 | Allow $\frac{1}{2}\int e^{4\theta} \, d\theta$<br><br>In their answer<br>May be implied |

| | | | | |
|---|---|---|---|---|
| 5 | (i) | Use correct definitions of cosh and sinh | B1 | |
| | | Attempt to square and subtract | M1 | On their definitions |
| | | Clearly get A.G. | A1 | |
| | | Show division by $\cosh^2$ | B1 | Or clear use of first result |
| | (ii) | Rewrite as quadratic in sech and attempt to solve | M1 | Or quadratic in cosh |
| | | Eliminate values outside $0 < \text{sech} \le 1$ | B1 | Or eliminate values outside $\cosh \ge 1$ (allow positive) |
| | | Get $x = \ln(2+\sqrt{3})$ | A1 | |
| | | Get $x = -\ln(2+\sqrt{3})$ or $\ln(2-\sqrt{3})$ | A1 | |
| 6 | (i) | Attempt at correct form of P.F. | M1 | Allow $Cx/(x^2+1)$ here; not $C = 0$ |
| | | Rewrite as 4= $A(1 + x)(1 + x^2) + B(1 - x)(1 + x^2) + (Cx + D)(1 - x)(1 + x)$ | M1 √ | From their P.F. |
| | | Use values of $x$/equate coefficients | M1 | |
| | | Get $A = 1$, $B = 1$ | A1 | cwo |
| | | Get $C = 0$, $D = 2$ | A1 | |
| | | | | SC Use of cover-up rule for $A,B$ M1<br>If both correct A1 cwo |
| | (ii) | Get $A\ln(1 + x) - B\ln(1 - x)$ | M1 | Or quote from List of Formulae |
| | | Get $D\tan^{-1}x$ | B1 | |
| | | Use limits in their integrated expressions | M1 | |
| | | Clearly get A.G. | A1 | |
| 7 | (i) | LHS = sum of areas of rectangles, area = 1x $y$-value from $x = 1$ to $x = n$ | B1 | |
| | | RHS = Area under curve from $x = 0$ to $n$ | B1 | |
| | (ii) | Diagram showing areas required | B1 | |
| | | Use sum of areas of rectangles | B1 | |
| | | Explain/show area inequality with limits in integral clearly specified | B1 | |
| | (iii) | Attempt integral as $kx^{4/3}$ | M1 | |
| | | Limits gives 348(.1) and 352(.0) | A1 | Allow one correct |
| | | Get 350 | A1 | From two correct values only |

| | | | | |
|---|---|---|---|---|
| **8** | **(i)** | Get $x = 1$, $y = 0$ | B1,B1 | |
| | **(ii)** | Rewrite as quadratic in $x$ | M1 | $(x^2y - x(2y + k) + y = 0)$ |
| | | Use $b^2 - 4ac \geq 0$ for all real $x$ | M1 | Allow >, = here |
| | | Get correct inequality | A1 | $4ky + k^2 \geq 0$ |
| | | State use of $k$>0 to A.G. | A1 | |
| | | | | SC Use differentiation (parts **(ii)** and **(iii)**) |
| | | | | Attempt prod/quotient rule M1 |
| | | | | Solve = 0 for $x = -1$ A1 |
| | | | | Use $x = -1$ only (reject $x$=1), $y = -\frac{1}{4}k$ A1 |
| | | | | Fully justify minimum B1 |
| | | | | Attempt to justify for all $x$ M1 |
| | | | | Clearly get A.G. A1 |
| | **(iii)** | Replace $y = -\frac{1}{4}k$ in quadratic in $x$ | M1 | |
| | | Get $x = -1$ only | A1 | |
| | |  | B1 | Through origin with minimum at $(-1, -\frac{1}{4}k)$ seen or given in the answer |
| | | | B1 | Correct shape (asymptotes and approaches) |
| | | | | **SC** (Start again) |
| | | | | Differentiate and solve d$y$/d$x$ = 0 for at least one $x$-value, independent of $k$ M1 |
| | | | | Get $x = -1$ only A1 |
| **9** | **(i)** | Rewrite tanh $y$ as $(e^y - e^{-y})/(e^y + e^{-y})$ | B1 | Or equivalent |
| | | Attempt to write as quadratic in $e^{2y}$ | M1 | |
| | | Clearly get A.G. | A1 | |
| | **(ii)** | **(a)** Attempt to diff. and solve = 0 | M1 | |
| | | Get tanh $x = b/a$ | A1 | |
| | | Use $(-1) < \tanh x < 1$ to show $b < a$ | B1 | |
| | | | | SC Use exponentials M1 |
| | | | | Get $e^{2x} = (a + b)/(a - b)$ A1 |
| | | | | Use $e^{2x} > 0$ to show $b < a$ B1 |
| | | | | SC Write $x = \tanh^{-1}(b/a)$ M1 |
| | | | | $= \frac{1}{2}\ln((1 + b/a)/(1 - b/a))$ A1 |
| | | | | Use ( ) > 0 to show $b < a$ B1 |
| | | **(b)** Get tanh $x$ =1/$a$ from part **(ii)(a)** | B1 | |
| | | Replace as ln from their answer | M1 | |
| | | Get $x = \frac{1}{2}\ln((a + 1)/(a - 1))$ | A1 | |
| | | Use $e^{\frac{1}{2}\ln((a+1)/(a-1))} = \sqrt{((a + 1)/(a - 1))}$ | M1 | At least once |
| | | Clearly get A.G. | A1 | |
| | | Test for minimum correctly | B1 | |
| | | | | **SC** Use of $y = \cosh x(a - \tanh x)$ and $\cosh x = 1/\text{sech}x = 1/\sqrt{(1 - \tanh^2 x)}$ |