OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS

4726

Further Pure Mathematics 2

Candidates answer on the Answer Booklet

OCR Supplied Materials:
- 8 page Answer Booklet
- List of Formulae (MF1)

Other Materials Required:
None

Friday 9 January 2009
Morning

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- This document consists of **8** pages. Any blank pages are indicated.

1 **(i)** Write down and simplify the first three terms of the Maclaurin series for e^{2x}. **[2]**

(ii) Hence show that the Maclaurin series for

$$\ln(e^{2x} + e^{-2x})$$

begins $\ln a + bx^2$, where a and b are constants to be found. **[4]**

2 It is given that α is the only real root of the equation $x^5 + 2x - 28 = 0$ and that $1.8 < \alpha < 2$.

(i) The iteration $x_{n+1} = \sqrt[5]{28 - 2x_n}$, with $x_1 = 1.9$, is to be used to find α. Find the values of x_2, x_3 and x_4, giving the answers correct to 7 decimal places. **[3]**

(ii) The error e_n is defined by $e_n = \alpha - x_n$. Given that $\alpha = 1.891\,574\,9$, correct to 7 decimal places, evaluate $\frac{e_3}{e_2}$ and $\frac{e_4}{e_3}$. Comment on these values in relation to the gradient of the curve with equation $y = \sqrt[5]{28 - 2x}$ at $x = \alpha$. **[3]**

3 **(i)** Prove that the derivative of $\sin^{-1} x$ is $\frac{1}{\sqrt{1 - x^2}}$. **[3]**

(ii) Given that

$$\sin^{-1} 2x + \sin^{-1} y = \tfrac{1}{2}\pi,$$

find the exact value of $\frac{dy}{dx}$ when $x = \frac{1}{4}$. **[4]**

4 **(i)** By means of a suitable substitution, show that

$$\int \frac{x^2}{\sqrt{x^2 - 1}}\, dx$$

can be transformed to $\int \cosh^2 \theta \, d\theta$. **[2]**

(ii) Hence show that $\int \frac{x^2}{\sqrt{x^2 - 1}}\, dx = \frac{1}{2}x\sqrt{x^2 - 1} + \frac{1}{2}\cosh^{-1} x + c$. **[4]**

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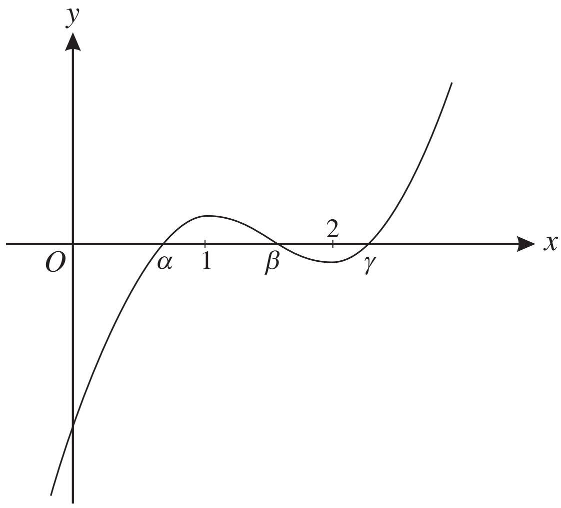

The diagram shows the curve with equation $y = f(x)$, where

$$f(x) = 2x^3 - 9x^2 + 12x - 4.36.$$

The curve has turning points at $x = 1$ and $x = 2$ and crosses the x-axis at $x = \alpha$, $x = \beta$ and $x = \gamma$, where $0 < \alpha < \beta < \gamma$.

(i) The Newton-Raphson method is to be used to find the roots of the equation $f(x) = 0$, with $x_1 = k$.

(a) To which root, if any, would successive approximations converge in each of the cases $k < 0$ and $k = 1$? **[2]**

(b) What happens if $1 < k < 2$? **[2]**

(ii) Sketch the curve with equation $y^2 = f(x)$. State the coordinates of the points where the curve crosses the x-axis and the coordinates of any turning points. **[4]**

6 **(i)** Using the definitions of $\cosh x$ and $\sinh x$ in terms of e^x and e^{-x}, show that

$$1 + 2\sinh^2 x \equiv \cosh 2x.$$ **[3]**

(ii) Solve the equation

$$\cosh 2x - 5\sinh x = 4,$$

giving your answers in logarithmic form. **[5]**

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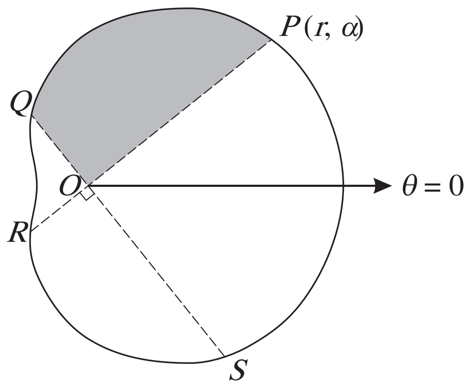


The diagram shows the curve with equation, in polar coordinates,

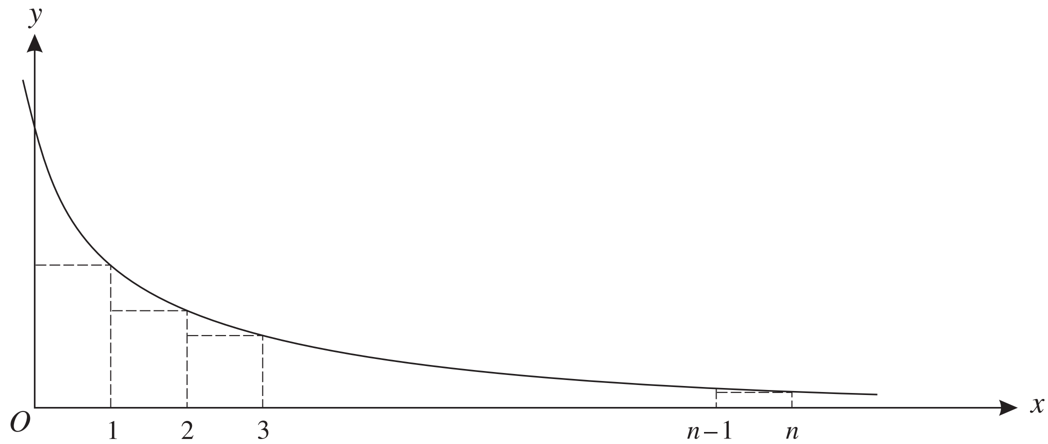
$$r = 3 + 2\cos\theta, \quad \text{for } 0 \leqslant \theta < 2\pi.$$

The points P, Q, R and S on the curve are such that the straight lines POR and QOS are perpendicular, where O is the pole. The point P has polar coordinates (r, α).

(i) Show that $OP + OQ + OR + OS = k$, where k is a constant to be found. **[3]**

(ii) Given that $\alpha = \frac{1}{4}\pi$, find the exact area bounded by the curve and the lines OP and OQ (shaded in the diagram). **[5]**

8



The diagram shows the curve with equation $y = \dfrac{1}{x+1}$. A set of n rectangles of unit width is drawn, starting at $x = 0$ and ending at $x = n$, where n is an integer.

(i) By considering the areas of these rectangles, explain why

$$\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n+1} < \ln(n+1).$$ **[5]**

(ii) By considering the areas of another set of rectangles, show that

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} > \ln(n+1).$$ **[2]**

(iii) Hence show that

$$\ln(n+1) + \frac{1}{n+1} < \sum_{r=1}^{n+1} \frac{1}{r} < \ln(n+1) + 1.$$ **[2]**

(iv) State, with a reason, whether $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r}$ is convergent. **[2]**

9 A curve has equation

$$y = \frac{4x - 3a}{2(x^2 + a^2)},$$

where a is a positive constant.

(i) Explain why the curve has no asymptotes parallel to the y-axis. **[2]**

(ii) Find, in terms of a, the set of values of y for which there are no points on the curve. **[5]**

(iii) Find the exact value of $\displaystyle\int_a^{2a} \frac{4x - 3a}{2(x^2 + a^2)}\,dx$, showing that it is independent of a. **[5]**

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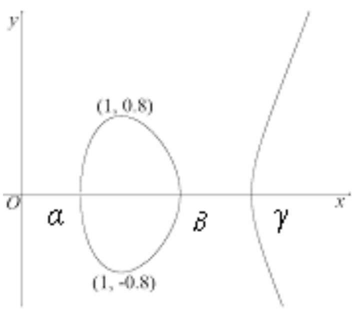
1	(i)	Give $1 + 2x + (2x)^2/2$ Get $1 + 2x + 2x^2$	M1 A1	Reasonable 3 term attempt e.g. allow $2x^2/2$ cao SC Reasonable attempt at f′(0) and f″(0) M1 Get $1+2x+2x^2$ cao A1
	(ii)	$\ln(\,(1+2x+2x^2\,)$ $+ (1-2x+2x^2)) =$ $\ln(2+4x^2) =$ $\ln2 + \ln(1 + 2x^2)$ $\ln2 + 2x^2$	M1 A1√ M1 A1	Attempt to sub for e^{2x} and e^{-2x} On their part **(i)** Use of log law in reasonable expression cao SC Use of Maclaurin for f ′(x) and f″(x) M1 One correct A1 Attempt f(0), f ′(0) and f″(0) M1 Get cao A1
2	(i)	$x_2 = 1.8913115$ $x_3 = 1.8915831$ $x_4 = 1.8915746$	B1 B1√ B1	x_2 correct; allow answers which round For any other from their working For all three correct
	(ii)	$e_3/e_2 = -\ 0.031(1)$ $e_4/e_3 = -0.036(5)$ State $f\,'(\alpha) \approx e_3/e_2 \approx e_4/e_3$	M1 A1 B1√	Subtraction and division on their values; allow ± Or answers which round to −0.031 and −0.037 Using their values but only if approx. equal; allow differentiation if correct conclusion; allow gradient for f′
3	(i)	Diff. $\sin y = x$ Use $\sin^2 + \cos^2 = 1$ to A.G. Justify +	M1 A1 B1	Implicit diff. to dy/dx = ±(1/cos y) Clearly derived; ignore ± e.g graph/ principal values
	(ii)	Get $2/(\sqrt{(1-4x^2)}$ $+ 1/(\sqrt{(1-y^2)}\ \mathrm{d}y/\mathrm{d}x = 0$ Find $y = \sqrt{3}/2$ Get $-2\sqrt{3}/3$	M1 A1 M1 A1√	Attempt implicit diff. and chain rule; allow e.g. $(1-2x^2)$ or $a/\sqrt{(1-4x^2)}$ Method leading to y AEEF; from their a above SC Write $\sin(\frac{1}{2}\pi - \sin^{-1}2x) = \cos(\sin^{-1}2x)$ B1 Attempt to diff. as above M1 Replace x in reasonable dy/dx and attempt to tidy M1 Get result above A1

4	**(i)**	Let $x=\cosh\theta$ such that $dx = \sinh\theta\, d\theta$	M1	
		Clearly use $\cosh^2 - \sinh^2 = 1$	A1	Clearly derive A.G.
	(ii)	Replace $\cosh^2\theta$	M1	Allow $a(\cosh 2\theta \pm 1)$
		Attempt to integrate their expression	M1	Allow $b\sinh 2\theta \pm a\theta$
		Get $\frac{1}{4}\sinh 2\theta + \frac{1}{2}\theta\ (+c)$	A1	
		Clearly replace for x to A.G.	B1	Condone no $+c$ SC Use expo. def^n; three terms M1 Attempt to integrate M1 Get $^1/_8(e^{2\theta}-e^{-2\theta}) + \frac{1}{2}\theta\ (+c)$ A1 Clearly replace for x to A.G. B1
5	**(i)**	(**a**) State (x=) α	B1	
		None of roots	B1	No explanation needed
		(**b**) Impossible to say	B1	
		All roots can be derived	B1	Some discussion of values close to 1 or 2 or central leading to correct conclusion
	(ii)		B1	Correct x for y=0; allow 0.591, 1.59, 2.31
			B1	Turning at (1,0.8) and/or (1,-0.8)
			B1	Meets x-axis at 90^0
			B1	Symmetry in x-axis; allow
6	**(i)**	Correct definitions used	B1	
		Attempt at $(e^x-e^{-x})^2/4 + 1$	M1	Allow $(e^x+e^{-x})^2+1$; allow /2
		Clearly derive A.G.	A1	
	(ii)	Form a quadratic in $\sinh x$	M1	
		Attempt to solve	M1	Factors or formula
		Get $\sinh x = -\frac{1}{2}$ or 3	A1	
		Use correct ln expression	M1	On their answer(s) seen once
		Get $\ln(-\frac{1}{2}+{}^{\sqrt{5}}/_2)$ and $\ln(3+\sqrt{10})$	A1	
7	**(i)**	OP= $3 + 2\cos\alpha$		
		OQ=$3 + 2\cos(\frac{1}{2}\pi+\alpha)$ $=3 - 2\sin\alpha$	M1	Any other unsimplified value
		Similarly OR=$3 - 2\cos\alpha$	M1	Attempt at simplification of at least two correct expressions
		OS=$3 + 2\sin\alpha$		
		Sum = 12	A1	cao
	(ii)	Correct formula with attempt at r^2	M1	Need not be expanded, but three terms if it is
		Square r correctly	A1	
		Attempt to replace $\cos^2\theta$ with $a(\cos 2\theta \pm 1)$	M1	
		Integrate their expression	A1√	Need three terms
		Get $^{11\pi}/_4 - 1$	A1	cao

8	(i)	Area = $\int 1/(x+1)\,dx$	B1	Include or imply correct limits
		Use limits to $\ln(n+1)$	B1	
		Compare area under curve to areas of rectangles	B1	Justify inequality
		Sum of areas = 1x(½ + ⅓ + …+ $1/(n+1)$)	M1	Sum seen or implied as 1 x y values
		Clear detail to A.G.	A1	Explanation required e.g. area of last rectangle at $x=n$, area under curve to $x=n$
	(ii)	Show or explain areas of rectangles above curve	M1	
		Areas of rectangles (as above) > area under curve	A1	First and last heights seen or implied; A.G.
	(iii)	Add 1 to both sides in **(i)** to make $\sum(^1/r)$	B1	Must be clear addition
		Add $^1/_{(n+1)}$ to both sides in **(ii)** to make $\sum(^1/r)$	B1	Must be clear addition; A.G.
	(iv)	State divergent	B1	Allow not convergent
		Explain e.g. $\ln(n+1)\to\infty$ as $n\to\infty$	B1	
9	(i)	Require denom. = 0	B1	
		<u>Explain</u> why denom. ≠ 0	B1	Attempt to solve, explain always > 0 etc.
	(ii)	Set up quadratic in x	M1	
		Get $2yx^2-4x+(2a^2y+3a)=0$	A1	
		Use $b^2 \geq 4ac$ for real x	M1	Produce quadratic inequality in y from their quad.; allow use of = or <
		Attempt to solve their inequality	M1	Factors or formula
		Get $y > {^1/_{2a}}$ and $y < {^{-2}/_a}$	A1	Justified from graph SC Attempt diff. by quot./product rule M1 Solve $dy/dx=0$ for two values of x M1 Get $x=2a$ and $x=-a/2$ A1 Attempt to find two y values M1 Get correct inequalities (graph used to justify them) A1
	(iii)	Split into two separate integrals	M1	
		Get $k\ln(x^2+a^2)$	A1	Or $p\ln(2x^2+2a^2)$
		Get $k_1\tan^{-1}(x/a)$	A1	k_1 not involving a
		Use limits and attempt to simplify	M1	
		Get $\ln 2.5 - 1.5\tan^{-1}2 + 3\pi/8$	A1	AEEF SC Sub. $x = a\tan\theta$ and $dx = a\sec^2\theta\,d\theta$ M1 Reduce to $\int p\tan\theta - p_1\,d\theta$ (ignore limits here) A1 Integrate to $p\ln(\sec\theta)-p_1\theta$ A1 Use limits (old or new) and attempt to simplify M1 Get answer above A1