RECOGNISING ACHIEVEMENT

ADVANCED GCE **4726/01**

MATHEMATICS

Further Pure Mathematics 2

WEDNESDAY 9 JANUARY 2008 Afternoon

Time: 1 hour 30 minutes

Additional materials: Answer Booklet (8 pages)
List of Formulae (MF1)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.
- **You are reminded of the need for clear presentation in your answers.**

This document consists of **4** printed pages.

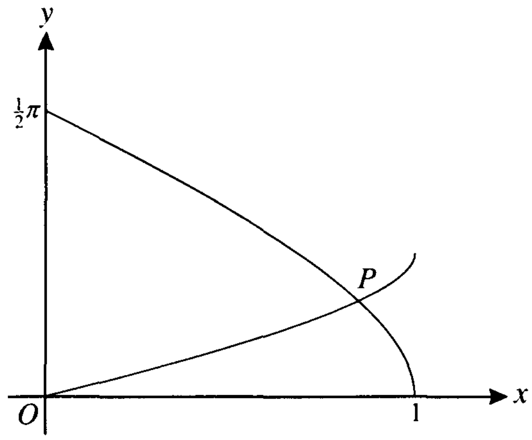
[Turn over

1 It is given that $f(x) = \ln(1 + \cos x)$.

(i) Find the exact values of $f(0)$, $f'(0)$ and $f''(0)$. [4]

(ii) Hence find the first two non-zero terms of the Maclaurin series for $f(x)$. [2]

2

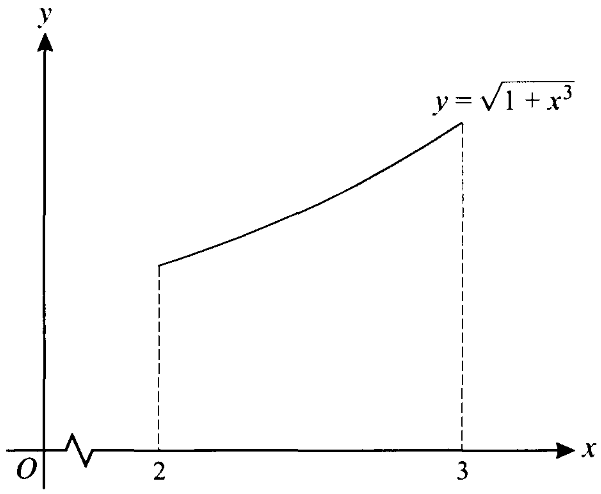


The diagram shows parts of the curves with equations $y = \cos^{-1} x$ and $y = \frac{1}{2}\sin^{-1} x$, and their point of intersection P.

(i) Verify that the coordinates of P are $\left(\frac{1}{2}\sqrt{3}, \frac{1}{6}\pi\right)$. [2]

(ii) Find the gradient of each curve at P. [3]

3



The diagram shows the curve with equation $y = \sqrt{1 + x^3}$, for $2 \leqslant x \leqslant 3$. The region under the curve between these limits has area A.

(i) Explain why $3 < A < \sqrt{28}$. [2]

(ii) The region is divided into 5 strips, each of width 0.2. By using suitable rectangles, find improved lower and upper bounds between which A lies. Give your answers correct to 3 significant figures. [4]

4 The equation of a curve, in polar coordinates, is

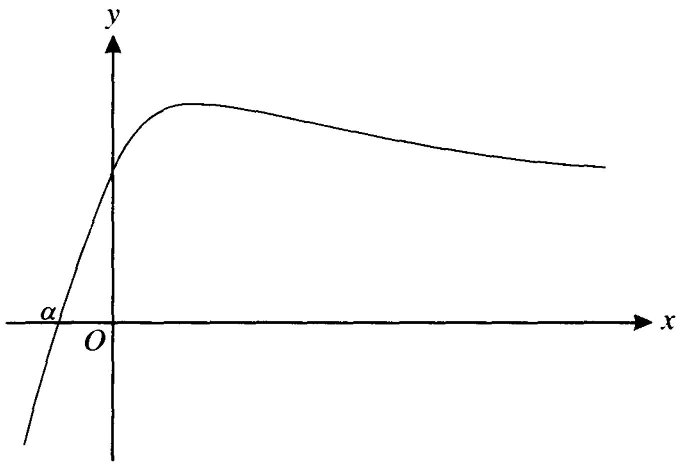
$$r = 1 + 2\sec\theta, \quad \text{for } -\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi.$$

(i) Find the exact area of the region bounded by the curve and the lines $\theta = 0$ and $\theta = \frac{1}{6}\pi$. [5]

[The result $\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta|$ may be assumed.]

(ii) Show that a cartesian equation of the curve is $(x-2)\sqrt{x^2+y^2} = x$. [3]

5



The diagram shows the curve with equation $y = xe^{-x} + 1$. The curve crosses the x-axis at $x = \alpha$.

(i) Use differentiation to show that the x-coordinate of the stationary point is 1. [2]

α is to be found using the Newton-Raphson method, with $f(x) = xe^{-x} + 1$.

(ii) Explain why this method will not converge to α if an initial approximation x_1 is chosen such that $x_1 > 1$. [2]

(iii) Use this method, with a first approximation $x_1 = 0$, to find the next three approximations x_2, x_3 and x_4. Find α, correct to 3 decimal places. [5]

6 The equation of a curve is $y = \dfrac{2x^2 - 11x - 6}{x - 1}$.

(i) Find the equations of the asymptotes of the curve. [3]

(ii) Show that y takes all real values. [5]

7 It is given that, for integers $n \geqslant 1$,

$$I_n = \int_0^1 \frac{1}{(1+x^2)^n}\,\mathrm{d}x.$$

(i) Use integration by parts to show that $I_n = 2^{-n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,\mathrm{d}x$. [3]

(ii) Show that $2nI_{n+1} = 2^{-n} + (2n-1)I_n$. [3]

(iii) Find I_2 in terms of π. [3]

8 **(i)** By using the definition of $\sinh x$ in terms of e^x and e^{-x}, show that

$$\sinh^3 x = \tfrac{1}{4}\sinh 3x - \tfrac{3}{4}\sinh x.$$ [4]

(ii) Find the range of values of the constant k for which the equation

$$\sinh 3x = k\sinh x$$

has real solutions other than $x = 0$. [3]

(iii) Given that $k = 4$, solve the equation in part **(ii)**, giving the non-zero answers in logarithmic form. [3]

9 **(i)** Prove that $\frac{\mathrm{d}}{\mathrm{d}x}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$. [3]

(ii) Hence, or otherwise, find $\int \frac{1}{\sqrt{4x^2-1}}\,\mathrm{d}x$. [2]

(iii) By means of a suitable substitution, find $\int \sqrt{4x^2-1}\,\mathrm{d}x$. [6]

4726 Further Pure Mathematics 2

1	**(i)**	Get $f'(x) = \pm \sin x/(1+\cos x)$	M1	Reasonable attempt at chain at any stage
		Get $f''(x)$ using quotient/product rule	M1	Reasonable attempt at quotient/product
		Get $f(0) = \ln 2$, $f'(0) = 0$, $f''(0) = -\frac{1}{2}$	B1	Any one correct from correct working
			A1	All three correct from correct working
	(ii)	Attempt to use Maclaurin correctly	M1	Using their values in $af(0)+bf'(0)x+cf''(0)x^2$; may be implied
		Get $\ln 2 - \frac{1}{4}x^2$	A1√	From their values; must be quadratic
2	**(i)**	Clearly verify in $y = \cos^{-1}x$	B1	i.e. $x=\frac{1}{2}\sqrt{3}$, $y=\cos^{-1}(\frac{1}{2}\sqrt{3})= {}^{1}/_{6}\pi$, or similar
		Clearly verify in $y = \frac{1}{2}\sin^{-1}x$	B1	Or solve $\cos y = \sin 2y$
			SR	Allow one B1 if not sufficiently clear detail
	(ii)	Write down at least one correct diff'al	M1	Or reasonable attempt to derive; allow ±
		Get gradient of –2	A1	cao
		Get gradient of 1	A1	cao
3	**(i)**	Get y- values of 3 and $\sqrt{28}$	B1	
		Show/explain areas of two rectangles equal y- value x 1, and relate to A	B1	Diagram may be used
	(ii)	Show $A>0.2(\sqrt{(1+2^3)} + \sqrt{(1+2.2^3)} + \ldots ..\sqrt{(1+2.83)})$	M1	Clear areas attempted below curve (5 values)
		$= 3.87(28)$	A1	To min. of 3 s.f.
		Show $A<0.2(\sqrt{(1+2.2^3)} + \sqrt{(1+2.4^3)} + \ldots \ldots + \sqrt{(1+3^3)})$	M1	Clear areas attempted above curve (5 values)
		$= 4.33(11) <4.34$	A1	To min. of 3 s.f.
4	**(i)**	Correct formula with correct r	M1	May be implied
		Expand r^2 as $A + B\sec\theta + C\sec^2\theta$	M1	Allow B = 0
		Get $C\tan\theta$	B1	
		Use correct limits in their answer	M1	Must be 3 terms
		Limits to ${}^{1}/_{12}\pi + 2\ln(\sqrt{3}) + {}^{2\sqrt{3}}/_{3}$	A1	AEEF; simplified
	(ii)	Use $x=r\cos\theta$ and $r^2 = x^2+y^2$	B1	Or derive polar form from given equation
		Eliminate r and θ	M1	Use their definitions
		Get $(x-2)\sqrt{(x^2+y^2)} = x$	A1	A.G.

5	(i)	Attempt use of product rule	M1	
		Clearly get $x = 1$	A1	Allow substitution of $x=1$
	(ii)	Explain use of tangent for next approx.	B1	Not use of G.C. to show divergence
		Tangents at successive approx. give $x>1$	B1	Relate to crossing x-axis; allow diagram
	(iii)	Attempt correct use of N-R with their derivative	M1	
		Get $x_2 = -1$	A1√	
		Get –0.6839, -0.5775, (-0.5672…)	A1	To 3 d.p. minimum
		Continue until correct to 3 d.p.	M1	May be implied
		Get –0.567	A1	cao
6	(i)	Attempt division/equate coeff.	M1	To lead to some $ax+b$ (allow $b=0$ here)
		Get $a = 2$, $b = -9$	A1	
		Derive/quote $x = 1$	B1	Must be equations
	(ii)	Write as quadratic in x	M1	$(2x^2-x(11+y)+(y-6)=0)$
		Use $b^2 \geq 4ac$ (for real x)	M1	Allow <, >
		Get $y^2 + 14y + 169 \geq 0$	A1	
		Attempt to justify positive/negative	M1	Complete the square/sketch
		Get $(y+7)^2 + 120 \geq 0$ – true for all y	A1	
			SC	Attempt diff; quot./prod. rule M1 Attempt to solve $dy/dx = 0$ M1 Show $2x^2 - 4x + 17 = 0$ has no real roots e.g. $b^2 - 4ac < 0$ A1 Attempt to use no t.p. M1 Justify all y e.g. consider asymptotes and approaches A1
7	(i)	Get $x(1+x^2)^{-n} - \int x.(-n(1+x^2)^{-n-1}.2x)\,dx$	M1	Reasonable attempt at parts
		Accurate use of parts	A1	
		Clearly get A.G.	B1	Include use of limits seen
	(ii)	Express x^2 as $(1+x^2) - 1$		
		Get $\dfrac{x^2}{(1+x^2)^{n+1}} = \dfrac{1}{(1+x^2)^n} - \dfrac{1}{(1+x^2)^{n+1}}$	B1	Justified
		Show $I_n = 2^{-n} + 2n(I_n - I_{n+1})$	M1	Clear attempt to use their first line above
		Tidy to A.G.	A1	
	(iii)	See $2I_2 = 2^{-1} + I_1$	B1	
		Work out $I_1 = \frac{1}{4}\pi$	M1	Quote/derive $\tan^{-1}x$
		Get $I_2 = \frac{1}{4} + \frac{1}{8}\pi$	A1	

8	**(i)**	Use correct exponential for sinh x	B1	
		Attempt to expand cube of this	M1	Must be 4 terms
		Correct cubic	A1	
		Clearly replace in terms of sinh	B1	(Allow RHS→ LHS or RHS = LHS separately)
	(ii)	Replace and factorise	M1	Or state sinh $x \neq 0$
		Attempt to solve for $\sinh^2 x$	M1	(= ¼(k-3)) or for k and use $\sinh^2 x>0$
		Get $k>3$	A1	Not ≥
	(iii)	Get $x = \sinh^{-1} c$	M1	($c= \pm\frac{1}{2}$); allow sinh $x = c$
		Replace in ln equivalent	A1√	As $\ln(\frac{1}{2}+\sqrt{\frac{5}{4}})$; their x
		Repeat for negative root	A1√	May be given as neg. of first answer (no need for x=0 implied)
			SR	Use of exponential definitions Express as cubic in $e^{2x} = u$ M1 Factorise to $(u-1)(u^2-3u+1)=0$ A1 Solve for x =0, $\frac{1}{2}\ln(\frac{3}{2} \pm \frac{\sqrt{5}}{2})$ A1
9	**(i)**	Get $\sinh y \frac{dy}{dx} =1$	M1	Or equivalent; allow ± Allow use of ln equivalent with Chain Rule
		Replace $\sinh y = \sqrt{(\cosh^2 y - 1)}$	A1	
		Justify positive grad. to A.G.	B1	e.g. sketch
	(ii)	Get $k \cosh^{-1} 2x$	M1	No need for c
		Get $k=\frac{1}{2}$	A1	
	(iii)	Sub. $x = k \cosh u$	M1	
		Replace all x to $\int k_1 \sinh^2 u \, du$	A1	
		Replace as $\int k_2(\cosh 2u - 1) \, du$	M1	Or exponential equivalent
		Integrate correctly	A1√	No need for c
		Attempt to replace u with x equivalent	M1	In their answer
		Tidy to reasonable form	A1	cao ($\frac{1}{2}x\sqrt{(4x^2 - 1)} - \frac{1}{4}\cosh^{-1} 2x$ (+c))