OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS (MEI) 4758/01

Differential Equations

Candidates answer on the Answer Booklet

OCR Supplied Materials:
- 8 page Answer Booklet
- MEI Examination Formulae and Tables (MF2)

Other Materials Required:
- Scientific or graphical calculator

Monday 24 May 2010
Afternoon

Duration: 1 hour 30 minutes

475801

INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer any **three** questions.
- Do **not** write in the bar codes.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g \text{ m s}^{-2}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

1 The equation of a curve in the x-y plane satisfies the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = 32x^2.$$

(i) Find the general solution. **[10]**

The curve has a minimum point at the origin.

(ii) Find the equation of the curve. **[4]**

(iii) Describe how the curve behaves for large negative values of x. **[2]**

(iv) Write down an approximate expression for y, valid for large positive values of x. **[1]**

(v) Sketch the curve. **[3]**

(vi) Use the differential equation to show that any stationary point below the x-axis must be a minimum. **[4]**

2 **(a)** **(i)** Find the general solution of

$$\frac{dy}{dt} + 2y = e^{-2t}.$$ **[6]**

(ii) Find the solution of

$$\frac{dz}{dt} + 2z = y,$$

where y is the general solution found in part **(i)**, subject to the conditions that $z = 1$ and $\frac{dz}{dt} = 0$ when $t = 0$. **[7]**

(b) The differential equation

$$\frac{dx}{dt} + 2x = \sin t$$

is to be solved.

(i) Find the complementary function and a particular integral. Hence state the general solution. **[6]**

(ii) Find the solution that satisfies the condition $\frac{dx}{dt} = 0$ when $t = 0$. **[3]**

(iii) Find approximate bounds between which x varies for large positive values of t. **[2]**

3 Water is leaking from a small hole near the base of a tank. The height of the surface of the water above the hole is y m at time t minutes.

(i) Consider first a cylindrical tank. The height of the water is modelled by the differential equation

$$\frac{dy}{dt} = -k\sqrt{y},$$

where k is a positive constant. The height of water is initially 1 m and after 2 minutes it is 0.81 m.

Find y in terms of t, stating the range of values of t for which the solution is valid. Sketch the solution curve. **[10]**

(ii) Now consider water leaking from a conical tank. The height of the water is modelled by the differential equation

$$\pi y^2 \frac{dy}{dt} = -0.4\sqrt{y}.$$

Find how long it takes the height to decrease from 1 m to 0.81 m. **[5]**

(iii) Now consider water leaking from a spherical tank. The height of the water is modelled by the differential equation

$$\pi(ay - y^2)\frac{dy}{dt} = -0.4\sqrt{y},$$

where a is the diameter of the sphere.

This equation is to be solved by Euler's method. The algorithm is given by $t_{r+1} = t_r + h$, $y_{r+1} = y_r + h\dot{y}_r$. The diameter is 2 m and initially the height is 1 m.

Use a step length of 0.1 to estimate the height after 0.2 minutes. **[5]**

(iv) For any tank, the velocity of the water leaving the hole is proportional to the square root of the height of the surface of the water above the hole.

By considering the rate of change of the volume of water, derive the differential equation

$$\frac{dy}{dt} = -k\sqrt{y}$$

for the cylindrical tank in part **(i)**. **[4]**

[Question 4 is printed overleaf.]

4 At time t, the quantities x and y are modelled by the simultaneous differential equations

$$\frac{dx}{dt} = 2x - 5y + 9e^{-2t},$$

$$\frac{dy}{dt} = x - 4y + 3e^{-2t}.$$

(i) Show that $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x = 3e^{-2t}$. **[5]**

(ii) Find the general solution for x. **[8]**

(iii) Find the corresponding general solution for y. **[4]**

Initially $x = 0$ and $y = 2$.

(iv) Find the particular solutions. **[4]**

(v) Describe the behaviour of the solutions as $t \to \infty$.

State, with reasons, whether this behaviour is different if the initial value of y is just less than 2, and the initial value of x is still 0. **[3]**

GCE

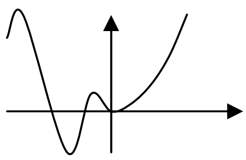
Mathematics (MEI)

Advanced GCE **4758**

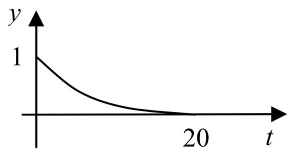
Differential Equations

Mark Scheme for June 2010

Oxford Cambridge and RSA Examinations

1(i)	$\alpha^2+4\alpha+8=0$	M1	Auxiliary equation	
	$\alpha=-2\pm 2\mathrm{j}$	A1		
	CF $\mathrm{e}^{-2x}(A\cos 2x+B\sin 2x)$	M1	CF for complex roots	
		F1	CF for their roots	
	PI $y=ax^2+bx+c$	B1		
	$\dot{y}=2ax+b, \ddot{y}=2a$			
	$2a+4(2ax+b)+8(ax^2+bx+c)=32x^2$	M1	Differentiate twice and substitute	
	$8a=32$	M1	Compare coefficients	
	$8a+8b=0$			
	$2a+4b+8c=0$	M1	Solve	
	$a=4, b=-4, c=1$	A1		
	GS $y=4x^2-4x+1+\mathrm{e}^{-2x}(A\cos 2x+B\sin 2x)$	F1	PI + CF with two arbitrary constants	10
(ii)	$x=0, y=0 \Rightarrow A=-1$	M1	Use condition	
	$y'=8x-4+\mathrm{e}^{-2x}(-2A\sin 2x+2B\cos 2x$ $-2A\cos 2x-2B\sin 2x)$	M1	Differentiate (product rule)	
	$x=0, y'=0 \Rightarrow 0=-4+(2B-2A) \Rightarrow B=1$	M1	Use condition	
	$y=4x^2-4x+1+\mathrm{e}^{-2x}(\sin 2x-\cos 2x)$	A1	Cao	4
(iii)	$x\to-\infty \Rightarrow y$ oscillates	B1	Oscillates	
	With (exponentially) growing amplitude	B1	Amplitude growing	2
(iv)	$y\sim(2x-1)^2$ or $4x^2-4x+1$	B1		1
(v)		B1	Minimum point at origin	
		B1	Oscillates for $x<0$ with growing amplitude	
		B1	Approximately parabolic for $x>0$	3
(vi)	At stationary point $\frac{\mathrm{d}y}{\mathrm{d}x}=0$	M1	Set first derivative (only) to zero in DE	
	So $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=32x^2-8y$	A1		
	$y<0 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}>0$	M1	Deduce sign of second derivative	
	$\Rightarrow$ minimum	E1	Complete argument	4

2(a)(i)	$IF = \exp\int 2\mathrm{d}t$	M1	Attempt IF	
	$= \mathrm{e}^{2t}$	A1		
	$\mathrm{e}^{2t}\dfrac{\mathrm{d}y}{\mathrm{d}t} + 2\mathrm{e}^{2t}y = 1$	M1*	Multiply by IF	
	$\dfrac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2t}y) = 1$	A1		
	$\mathrm{e}^{2t}y = t + A$	*M1A1	Integrate both sides	6
	$[y = \mathrm{e}^{-2t}(t + A)]$			
	Alternative method:			
	CF $y = E\mathrm{e}^{-2t}$	B1		
	PI $y = Ft\mathrm{e}^{-2t}$	B1		
	In DE: $\mathrm{e}^{-2t}(F - 2Ft) + 2Ft\mathrm{e}^{-2t} = \mathrm{e}^{-2t}$	M1		
	$F = 1$	M1A1		
	$y = \mathrm{e}^{-2t}(t + E)$	F1		
(ii)	$\dfrac{\mathrm{d}z}{\mathrm{d}t} + 2z = \mathrm{e}^{-2t}(t + A)$			
	$I = \mathrm{e}^{2t}$	B1	Correct or follows (i)	
	$\dfrac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{2t}z) = t + A$	M1	Multiply by IF and integrate	
	$\mathrm{e}^{2t}z = \frac{1}{2}t^2 + At + B$	A1		
	$z = \mathrm{e}^{-2t}(\frac{1}{2}t^2 + At + B)$			
	$t = 0, z = 1 \Rightarrow 1 = B$	M1	Use condition	
	$\dot{z} = -2\mathrm{e}^{-t}(\frac{1}{2}t^2 + At + B) + \mathrm{e}^{-2t}(t + A)$	M1	Differentiate (product rule)	
	$t = 0, \dot{z} = 0 \Rightarrow 0 = -2B + A \Rightarrow A = 2$	M1	Use condition	
	$z = \mathrm{e}^{-2t}(\frac{1}{2}t^2 + 2t + 1)$	A1		7
	Alternative method:			
	PI $x = (Pt + Qt^2)\mathrm{e}^{-2t}$	B1	Correct form of PI	
	$P = A$ and $Q = 0.5$			
	$z = \mathrm{e}^{-2t}(\frac{1}{2}t^2 + At + B)$	M1A1	Complete method	
	Then as above			
(b)(i)	$\alpha + 2 = 0 \Rightarrow \alpha = -2$			
	CF $x = C\mathrm{e}^{-2t}$	B1	CF correct	
	PI $x = a\sin t + b\cos t$	B1	Correct form of PI	
	$\dot{x} = a\cos t - b\sin t$			
	In DE: $a\cos t - b\sin t + 2a\sin t + 2b\cos t = \sin t$	M1	Differentiate and substitute	
	$a + 2b = 0, -b + 2a = 1$	M1	Compare and solve	
	$\Rightarrow a = \frac{2}{5}, b = -\frac{1}{5}$	A1		
	GS $x = \frac{1}{5}(2\sin t - \cos t) + C\mathrm{e}^{-2t}$	F1	Their PI + CF	6
(ii)	$\dot{x} = 0, t = 0 \Rightarrow x = 0$ (from DE)	M1	Or differentiate	
	$0 = -\frac{1}{5} + C$	M1	Use condition	
	$x = \frac{1}{5}(2\sin t - \cos t + \mathrm{e}^{-2t}$	A1		3
(iii)	For large t, $x \approx \frac{1}{5}(2\sin t - \cos t) = \frac{1}{5}\sqrt{5}\sin(t\text{-}\phi)$	M1	Complete method	
	So x varies between $-\frac{1}{5}\sqrt{5}$ and $\frac{1}{5}\sqrt{5}$	A1	Accept $\lvert x\rvert \le \frac{1}{5}\sqrt{5}$	2

3(i)	$\int y^{-\frac{1}{2}} \mathrm{d}y = \int -k \mathrm{d}t$	M1	Separate and integrate	
	$2y^{\frac{1}{2}} = -kt + B$	A1	LHS	
		A1	RHS	
	$t = 0, y = 1 \Rightarrow 2 = B$	M1	Use condition	
	$t = 2, y = 0.81 \Rightarrow 1.8 = -2k + 2$	M1	Use condition	
	$\Rightarrow k = 0.1$	A1		
	$y^{\frac{1}{2}} = 1 = 0.05t$			
	$y = (1 - 0.05t)^2$	A1		
	Valid for $1 - 0.05t \geq 0$, i.e. $t \leq 20$	B1√	√ on arithmetical error in k	
		B1 B1	Shape Intercepts	
				10
(ii)	$\int \pi y^{\frac{3}{2}} \mathrm{d}y = \int -0.4 \mathrm{d}t$	M1	Separate and integrate	
	$\frac{2}{5}\pi y^{\frac{5}{2}} = -0.4t + C$	A1	LHS	
		A1	RHS	
	$t = 0, y = 1 \Rightarrow C = \frac{2}{5}\pi$	M1	Use condition	
	$y = 0.81 \Rightarrow t = 1.287$	A1		5
(iii)	$\dot{y} = -\frac{0.4\sqrt{y}}{\pi(2y - y^2)}$	M1	Rearrange (implied by correct values)	
	t y $\dot{y}$ $h\dot{y}$	M1	Use algorithm	
	0 1 –0.12732 –0.01273	A1	$y(0.1)$ (awrt 0.987)	
	0.1 0.987268 –0.12653 –0.01265	M1	Use algorithm	
	0.2 0.974614	A1	$y(0.2)$ (0.974 to 0.975)	5
(iv)	If $V =$ volume, $v =$ velocity, $A =$ horizontal cross-sectional area,			
	then $\frac{\mathrm{d}V}{\mathrm{d}t} = -k_1 v$	M1	Rate of change of volume	
	$v = k_2\sqrt{y}$			
	$A\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{dV}}{\mathrm{d}t}$	M1	Relate rates of change of y and volume	
	$\Rightarrow A\frac{\mathrm{d}y}{\mathrm{d}t} = -k_1 k_2\sqrt{y}$	M1	Eliminate volume and/or velocity	
	$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = -k\sqrt{y}$	E1	Complete argument	4

4(i)	$5y = 2x + 9e^{-2t} - \dot{x}$	M1	y or $5y$ in terms of $x, \dot{x}$	
	$5\dot{y} = 2\dot{x} - 18e^{-2t} - \ddot{x}$	M1	Differentiate	
	$\frac{1}{5}(2\dot{x} - 18e^{-2t} - \ddot{x})$	M1	Substitute for y	
	$= x - \frac{4}{5}(2x + 9e^{-2t} - \dot{x}) + 3e^{-2t}$	M1	Substitute for $\dot{y}$	
	$\Rightarrow \ddot{x} + 2\dot{x} - 3x = 3e^{-2t}$	E1		5
(ii)	$\alpha^2 + 2\alpha - 3 = 0$	M1	Auxiliary equation	
	$\Rightarrow \alpha = 1, -3$	A1		
	CF $Ae^{t} + Be^{-3t}$	F1	CF for their roots	
	PI $x = ae^{-2t}$	B1	PI of correct form	
	$\dot{x} = -2ae^{-2t}, \ddot{x} = 4ae^{-2t}$	M1	Differentiate and substitute	
	$(4a - 4a - 3a)e^{-2t} = 3e^{-2t}$	M1	Compare coefficients and solve	
	$a = -1$	A1		
	GS $x = Ae^{t} + Be^{-3t} - e^{-2t}$	F1	PI + CF with two arbitrary constants	8
(iii)	$y = \frac{1}{5}(2x + 9e^{-2t} - \dot{x})$	M1		
	$\frac{1}{5}(2Ae^{t} + 2Be^{-3t} - 2e^{-2t} + 9e^{-2t}$	M1	Differentiate and substitute	
	$-(Ae^{t} - 3Be^{-3t} + 2e^{-2t}))$	F1	Expression for $\dot{x}$ follows their GS	
	$y = \frac{1}{5}Ae^{t} + Be^{-3t} + e^{-2t}$	A1		4
(iv)	$t = 0, x = 0 \Rightarrow 0 = A + B - 1$	M1	Use condition	
	$t = 0, y = 2 \Rightarrow 2 = \frac{1}{5}A + B + 1$	M1	Use condition	
	$\Rightarrow A = 0, B = 1$			
	$x = e^{-3t} - e^{-2t}$	A1		
	$y = e^{-3t} + e^{-2t}$	A1		4
(v)	As $t \to \infty, x \to 0, y \to 0$	B1		
	$y(0) < 2 \Rightarrow A > 0$	M1	Consider coefficient(s) of e^{t} and mention of $y < 2$	
	$x, y \to \infty$ as $t \to \infty$	E1	Complete argument	3