OCR RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2615

Statistics 3

Wednesday **21 JANUARY 2004** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer **all** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 4 printed pages.

1 The continuous random variable X has probability density function $f(x)$ given by

$$f(x) = kx, \qquad 0 < x \leqslant 5.$$

(i) Show that $k = \frac{2}{25}$. [2]

(ii) Find the cumulative distribution function of X. [2]

(iii) Show that observations from X fall in the intervals

$$0 < x \leqslant 1, \quad 1 < x \leqslant 2, \quad 2 < x \leqslant 3, \quad 3 < x \leqslant 4, \quad 4 < x \leqslant 5$$

in the proportions 1 : 3 : 5 : 7 : 9. [4]

As part of a large simulation, a computer program is required to generate observations from X. In a random sample of 200 such observations, the frequencies falling in the intervals in part (iii) are as follows.

Interval	$0<x\leqslant 1$	$1<x\leqslant 2$	$2<x\leqslant 3$	$3<x\leqslant 4$	$4<x\leqslant 5$
Observed frequency	8	26	41	51	74

(iv) Use a χ^2 test at the 5% level of significance to confirm that the observations from the program may reasonably be assumed to follow the required proportions. [6]

[Total 14]

2 Bottles of wine are supposed to contain at least 750 ml. At a filling plant, the volume of wine delivered by a machine to a bottle is a Normally distributed random variable with mean 754 ml and standard deviation 2 ml.

(i) Find the probability that, on a randomly chosen occasion, the volume of wine delivered to a bottle is at least 750 ml. [2]

(ii) Find the probability that the total volume of wine delivered by the machine on 12 randomly chosen occasions exceeds 9030 ml. [3]

The bottles are manufactured by an independent process. The capacities of the bottles are given by the Normally distributed random variable with mean 760 ml and standard deviation 3 ml.

(iii) Find the probability that a randomly chosen bottle has enough capacity for the amount of wine delivered to it. [4]

The manufacturing process for the bottles is changed with the intention of increasing their capacities. The change also affects the standard deviation. An inspector takes a random sample of 9 such bottles after they have passed through the filling plant, and carefully measures the excess capacity of each, i.e. the value of $b - w$ where b is the capacity of the bottle and w is the volume of wine in it. The values of $b - w$ are as follows, in ml.

3.1 7.6 7.2 5.8 1.2 13.4 7.1 10.2 6.8

(iv) Obtain a two-sided 95% confidence interval for the population mean excess capacity. Use your interval to comment on whether the manufacturing process for the bottles appears to have been effective. [6]

[Total 15]

3 A factory manager is specifying a new storage tank for a particular chemical. In routine use, the tank will be filled to capacity each weekend. There should be enough chemical to last until the next weekend, as emergency deliveries are very expensive. On the other hand, money is wasted if an excessive amount of the chemical is stored.

The volume of chemical required varies from week to week and is modelled by a Normally distributed random variable X. The manager is investigating the mean of X. Data are available for a random sample of 15 weeks, giving the volumes of the chemical used in each week. These are as follows (in litres).

1962 1928 1943 1939 1866 1964 1942 1996 1909 1940 1897 1924 1978 1944 1992

The standard deviation of X is taken from long experience to be 28 litres.

(i) A 2000-litre tank will be specified if the mean of X is no more than 1930 litres. Carry out a 5% significance test to examine whether a 2000-litre tank should be specified, stating clearly the null and alternative hypotheses and the conclusion. [9]

(ii) Write down the probability of a Type I error for the test and calculate the probability of a Type II error if in fact the mean of X is 1958 litres. [6]

[Total 15]

4 [You may in this question use the results $\int_0^\infty e^{-ky}dy = \frac{1}{k}$ and $\int_0^\infty ye^{-ky}dy = \frac{1}{k^2}$.]

A chain of furniture stores orders chairs of a particular design from a supplier. Sometimes the supplier is able to supply the chairs from its own warehouse, but on other occasions the chairs have to be made to order. The waiting time (in days) for the store to receive a chair from the supplier is modelled by the continuous random variable T having probability density function $f(t)$ given by

$$f(t) = \tfrac{1}{20}e^{-t/10} + \tfrac{1}{40}e^{-t/20}, \qquad t > 0.$$

(i) Verify that this is a valid probability density function. [4]

(ii) Show that the mean waiting time is 15 days. [3]

You are given that the variance of the waiting time (measured in days squared) is 275.

(iii) Find the probability that the average waiting time for 30 orders, considered as a random sample, exceeds 21 days. [4]

The store management requests the supplier to reduce both the mean and the variance of the waiting time. The management and the supplier initially agree that, while there might still be wide variations in individual waiting times, the mean waiting time for random samples of 30 orders, denoted by θ, should be reduced to below 14.

(iv) Subsequently, the average waiting time, x, for a random sample of 30 orders is noted on 6 separate occasions and found to be as follows.

10.9 13.6 14.2 7.6 17.2 13.8

[For information: for these data, $\bar{x}$ is 12.883 and s (defined with divisor $n - 1$) is 3.273.]

Test at the 5% level of significance the null hypothesis that $\theta = 14$ against the alternative $\theta < 14$. [5]

[Total 16]

Mark Scheme

Q1	$f(x) = kx,\ 0 < x \le 5$.			
(i)	$1 = \int_0^5 kx\,dx$ $= \left[\frac{kx^2}{2}\right]_0^5 = k\frac{25}{2}$ $\therefore k = \frac{2}{25}$	M1 A1	Set up requirement, including "= 1". Use of cdf requires sight of "F(5) = 1" or equivalent. c.a.o. Convincingly shown; ANSWER GIVEN.	2
(ii)	Cdf is $F(x) = \int_0^x \frac{2t}{25}dt$ $= \left[\frac{t^2}{25}\right]_0^x = \frac{x^2}{25}$ (for $0 < x \le 5$)	M1 A1	Definition of cdf, including limits, possibly implied later. c.a.o. Condone the omission of the domain.	2
(iii)	Using cdf (or by longer methods via the pdf), $P(0 < x \le 1) = F(1) = \frac{1}{25}$ $P(1 < x \le 2) = F(2) - F(1) = \frac{4}{25} - \frac{1}{25} = \frac{3}{25}$ and similarly $P(2 < x \le 3) = \frac{5}{25}$, $P(3 < x \le 4) = \frac{7}{25}$, $P(4 < x \le 5) = \frac{9}{25}$ i.e. proportions are 1 : 3 : 5 : 7 : 9	M1 A1 A1 A1	 For any three correct. For all five correct. BEWARE PRINTED ANSWER.	4
(iv)	Obs f's 8 26 41 51 74 Exp f's 8 24 40 56 72 $X^2 = \frac{(8-8)^2}{8} + \frac{(26-24)^2}{24} + \ldots$ $= 0 + 0{\cdot}1667 + 0{\cdot}025 + 0{\cdot}4464 + 0{\cdot}0556$ $= 0{\cdot}69(365)$ Refer to χ^2_4. Upper 5% point 9·488. Not significant. Seems observations are from this distribution.	M1 A1 M1 A1 A1 A1	Award even if X^2 is wrong. Accept anything that implies use of χ^2_4. No ft from here if wrong. No ft if not upper 5% point. ft only c's test statistic. ft only c's test statistic. References to <u>model</u> or fit not acceptable here. Must not be too assertive. <u>Special case:</u> χ^2_5 and 11·07 can get either (not both) of these final two marks.	6
				14

Q2	W = volume of wine ~ N(754, 2^2) B = capacity of bottle ~ N(760, 3^2)			
(i)	$P(W > 750) = P(N(0, 1) > \frac{750-754}{2} = -2)$ $= 0{\cdot}9772$	M1 A1	For standardising. Award once, here or elsewhere.	 2
(ii)	$W_1 + W_2 + \ldots + W_{12} \sim N(9048,$ $2^2 + 2^2 + \ldots + 2^2 = 48)$ $\therefore$ P(Total > 9030) = $P(N(0, 1) > \frac{-18}{\sqrt{48}} = -2{\cdot}598)$ $= 0{\cdot}9953$	B1 B1 A1	Mean. Variance. Accept sd = √48. c.a.o.	 3
(iii)	Want $P(B - W > 0)$ $B - W \sim N(6,$ $3^2 + 2^2 = 13)$ $\therefore$ want $P(N(6, 13) > 0) = P(N(0, 1) > \frac{-6}{\sqrt{13}} = -1{\cdot}664)$ $= 0{\cdot}9519$	M1 B1 B1 A1	Allow $P(B - W < 0)$ provided probability of complement found eventually. Mean. Variance. Accept sd = √13. c.a.o.	 4
(iv)	$\bar{x} = 6{\cdot}93(33)$, $s_{n-1} = 3{\cdot}56(896)$ ($s_{n-1}^2 = 12{\cdot}7375$) CI is given by 6·93(33) ± 2·306 $\times \frac{3{\cdot}56896}{\sqrt{9}}$ = 6·93(33) ± 2·74(33) = (4·19, 9·67 or 9·68) This interval is <u>well</u> away from zero and <u>suggests</u> that the bottle manufacturing process is OK.	B1 M1 B1 M1 A1 E1	Allow $s_n = 3{\cdot}36(485)$ ($s_n^2 = 11{\cdot}3222$) only if correctly used in sequel. ft c's $\bar{x}$ ±. From t_8. ft c's s_{n-1}. Allow c's s_n only with $\sqrt{8}$ and vice-versa. c.a.o. Must be expressed as an interval. Accept any other sensible comment.	 6
				15

Q3				
(i)	$H_0 : \mu = 1930$ $H_1 : \mu > 1930$	B1 B1	Allow "μ" to be undefined (as it is essentially given by context in the question), but do NOT allow "$\bar{X} = \ldots$" or similar unless $\bar{X}$ is clearly and explicitly stated to be a population mean. Hypotheses in words only must include "population".	
	$n = 15$, $\Sigma x = 29124$, $\bar{x} = 1941 \cdot 6$	B1	$\sigma = 28$ is given.	
	Test statistic is $\dfrac{1941{\cdot}6 - 1930}{\left(\dfrac{28}{\sqrt{15}}\right)}$	M1	ft c's $\bar{x}$. Allow alternative: $1930 + (\text{c's } 1{\cdot}645) \times \frac{28}{\sqrt{15}}$ (=1941·89) for subsequent comparison with $\bar{x}$. (Or $\bar{x} - (\text{c's } 1{\cdot}645) \times \frac{28}{\sqrt{15}}$ (= 1929·71) for comparison with 1930.)	
	$= 1{\cdot}60(45)$	A1	c.a.o. (but ft from here if this is wrong). Use of $1930 - \bar{x}$ scores M1A0, but ft (must have lower 5% point if that mark is to be given).	
	Refer to N(0, 1). Upper 5% point is 1·645. Not significant. Reasonable to accept that a 2000-litre tank should be specified.	M1 A1 A1 E1	No ft from here if wrong. No ft from here if wrong. $\Phi(1{\cdot}6045) = 0{\cdot}9456$ or $0{\cdot}9457$. ft only c's test statistic. ft only c's test statistic.	9
(ii)	P(Type I error) = 0.05 If $\mu = 1958$, $\bar{X} \sim N(1958, \frac{28^2}{15})$	B1 M1	Accept "5%". For the distribution of $\bar{X}$ with $\mu = 1958$.	
	H_0 is accepted if $\bar{X} < 1930 + 1{\cdot}645 \times \frac{28}{\sqrt{15}}$ $= 1941{\cdot}89(26)$	M1	For the critical point of the test.	
	So P(Type II error) = $P(N(1958, \frac{28^2}{15}) < 1941{\cdot}89)$	M1	But M0 if RHS = 1930 or 1941·6.	
	$= P(N(0, 1) < -2{\cdot}228)$ $= 0{\cdot}0129$	M1 A1	Standardising. c.a.o.	6
				15

Q4	Given: $\int_0^\infty e^{-ky}dy = \frac{1}{k}$, $\int_0^\infty ye^{-ky}dy = \frac{1}{k^2}$ $f(t) = \frac{1}{20}e^{-t/10} + \frac{1}{40}e^{-t/20}$, $t > 0$			
(i)	Verification of $f(t) \geq 0$. $\int_0^\infty f(t)dt = \int_0^\infty (\frac{1}{20}e^{-t/10} + \frac{1}{40}e^{-t/20})dt$ $= \frac{1}{20}(10) + \frac{1}{40}(20)$ $= \frac{1}{2} + \frac{1}{2} = 1$	B1 M1 A1 A1	For attempt to integrate. For correct use of quoted results (or correct integration and use of limits).	4
(ii)	$\mu = \int_0^\infty tf(t)dt$ $= \int_0^\infty (\frac{1}{20}te^{-t/10} + \frac{1}{40}te^{-t/20})dt$ $= \frac{1}{20}(100) + \frac{1}{40}(400)$ $= 5 + 10 = 15$	M1 A1 A1	Must include correct limits. For correct use of quoted results. BEWARE PRINTED ANSWER. <u>Special case:</u> If M0 awarded for absence of limits but subsequent work correct then allow SC B1B1.	3
(iii)	$\bar{T} \sim N($ $15,$ $\frac{275}{30} = 9{\cdot}1667)$ $P(\bar{T} > 21) = P(N(0, 1) > \frac{6}{\sqrt{9{\cdot}1667}} = 1{\cdot}982)$ $= 0{\cdot}0238$	M1 B1 B1 A1	Normal distribution. Mean. Or 450 if $T_1 + \ldots + T_{30}$ used. Variance. Or 8250 if $T_1 + \ldots + T_{30}$ used. Or $P(T_1 + \ldots + T_{30} > 630) = \ldots$ c.a.o.	4
(iv)	Given: $n = 6$, $\bar{x} = 12{\cdot}883$, $s_{n-1} = 3{\cdot}273$ Test statistic is $\frac{12{\cdot}883 - 14}{\left(\frac{3{\cdot}273}{\sqrt{6}}\right)}$ $= -0{\cdot}836$ Refer to t_5. Lower 5% point is $-2{\cdot}015$. Not significant. No evidence that $\theta < 14$.	M1 A1 M1 A1 A1	Allow alternative: 14 + (c's − 2·015) × $\frac{3{\cdot}273}{\sqrt{6}}$ (= 11·308) for subsequent comparison with $\bar{x}$. (Or $\bar{x}$ − (c's − 2·015) × $\frac{3{\cdot}273}{\sqrt{6}}$ (= 15·575) for comparison with 14.) c.a.o. (but ft from here if this is wrong). Use of 14 − $\bar{x}$ scores M1A0, but ft (must have upper 5% point if that mark is to be given). No ft from here if wrong. No ft from here if wrong. ft only c's test statistic. Verbal statement in context must be present. <u>Special case:</u> <u>This</u> mark may be awarded if (t_6 and −1·943) or (t_5 and −2·571) used.	5
				16

Examiner’s Report

2615 Statistics 3

General Comments

There were 357 candidates from 63 centres (January 2003: 334 from 66). The general standard of the scripts seen was pleasing: many candidates were clearly well prepared for this paper. The quality of work and the distribution of marks scored were very similar to last January. However, as in the past, comments and explanation were a consistent weakness.

Invariably all four questions were attempted. The average marks scored on individual questions were broadly similar to each other, with Question 2 appearing slightly better than the others and Question 4 slightly worse.

Comments on Individual Questions

Question 1 (Continuous random variables and Chi-squared test; computer simulation)

(i) Almost all candidates answered this part easily and correctly.

(ii) It was disappointing to see just how many candidates could not find the cumulative distribution function satisfactorily. Quite clearly most knew (or guessed) what the answer should be but fewer were able to set up the correct integral with the correct limits. Limits were often either omitted altogether or shown as 0 and 5. Even when something appropriate was seen, it was common for there to be no distinction between the independent variable used for integration and the variable upper limit.

(iii) There were occasional "faked" attempts at this part of the question, but since (by one means or another) the c.d.f. was known, the examiner saw many correct answers which earned full marks. In fact some candidates wrote out quite lengthy and thorough answers.

(iv) As in the past, candidates showed that they are very competent at working out the test statistic for a Chi-squared test. There was rarely a problem over the correct number of degrees of freedom or the critical value of the test. However, many candidates stated their final conclusions in terms that either suggested that they had not paid sufficient attention to the detailed wording of the question, or that were considered to be much too assertive about what the test had revealed.

(ii) $F(x) = x^2/25$; (iv) test statistic 0·6937, critical value 9·488.

Question 2 (Combinations of Normal distributions; confidence interval for the population mean; filling wine bottles)

In this question there was evidence of many candidates making effective use of the built-in Normal distribution and other functions on their calculator. By and large candidates were competent at using the Normal distribution. However, there were many instances of poor presentation which left the examiner needing to infer the candidates' intentions. Also some candidates seem to have a very sloppy attitude to the need for quoting from tables as accurately as they can. Candidates who include a sketch as part of their answer (probably intended as an aid to themselves) are almost always successful.

(i) This part was usually correct.

(ii) This part too was usually correct. Occasionally the wrong variance of the distribution of the total volume of wine was used. On a matter of precision and notation, rather a lot of candidates appeared to be thinking in terms of $12W$ instead of $W_1 + \ldots + W_{12}$, even though they quoted the variance for the latter.

(iii) Once again, this part was usually correct. Any problems encountered were likely to be associated with the variance of the difference of the two distributions and/or with a correct formulation of the requirement of the question.

(iv) There were many correct answers for the confidence interval. It was pleasing to see so many candidates identify correctly the appropriate percentage point from the relevant t-distribution. The main area of difficulty was with the interpretation of the interval in relation to the effectiveness of the manufacturing process. "Effective" or "not effective" were insufficient responses; some additional explanation was required. Very many comments revealed incorrect understanding of a confidence interval to quite a worrying extent. Broadly two types of argument were considered acceptable, illustrated as follows: "The interval is well away from zero, suggesting that the process has been effective", or "This interval happens to contain the original population mean, suggesting that the process has not been effective."

(i) 0·9772; (ii) 0·9953; (iii) 0·9519; (iv) (4·19, 9·67).

Question 3 (Hypothesis test for the population mean using the Normal distribution; Type II errors; volumes of chemical used)

(i) The hypotheses were usually stated correctly; only occasionally did a candidate neglect to use the symbol μ. The correct test statistic was calculated most of the time, although a few candidates overlooked the fact that the variance was given in the question. Some attempted to use the t-distribution (with $\nu = 14$) for their critical value. Interestingly, however, there was no evidence of candidates being consistent about their use of a variance estimated from the data. The conclusions of the test were generally acceptable but once again revealed that some did not pay sufficient attention to the wording of the question.

(ii) Almost all knew the probability of the Type I error to be the same as the significance level of the test. The work presented for the rest of this part of the question showed more understanding of the nature of Type II errors than has been the case in the past. Even so there were many thoughtless and frustratingly unnecessary mistakes. Many candidates chose either the old population mean (1930) or the test statistic from part (i) (1941·6) as their critical point. The wrong side of the critical point was often identified with the type II error. The variance of the distribution was taken as 28^2 even though the correct value had been used in part (i).

(i) test statistic 1·6045, critical value 1·645; (ii) 0·05, critical point 1941·89, 0·0129

Question 4 (Continuous random variables; Central Limit Theorem; Hypothesis test for the population mean using the t distribution; delivery waiting times)

In the first two parts of this question the attention to correct notation for the definite integrals left much to be desired.

(i) From the entire entry for this paper fewer than about 10 candidates even considered the need for a probability density function to be positive. Of those candidates barely half actually said anything that amounted to a verification of it. Thereafter many offered decent demonstrations that the total area under the curve was equal to 1. Most used the relevant result given at the start of the question; some, with mixed success, integrated the p.d.f. for themselves.

(ii) Again many candidates knew what they needed to find and recognised the relevance of the given result. This time, however, since the required answer is given in the question, it was tempting for some to try to fiddle the outcome. Some tried to do the integration for themselves, but very few of them were up to the task of integration by parts.

(iii) There were many correct answers to this part of the question, but it was not always obvious that candidates really knew that they were using the Central Limit Theorem.

(iv) Good candidates had little difficulty in calculating the test statistic from the information provided. There was rather more difficulty over the correct distribution for the test and the correct percentage point from the tables. Furthermore it seemed that many candidates were not comfortable with a negative critical value as suggested by the alternative hypothesis and the test statistic. They can, of course, compare absolute values, but it was often difficult to decide whether that was being done by design or accident. The conclusion of this test was often spoiled by making no reference to the context of the problem or by being too assertive about the mean wasting time. Again a careful reading of the question would have provided a strong hint about the form of wording expected.

(iii) 0·0238; (iv) test statistic –0·836, critical value –2·015