OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2615

Statistics 3

Monday **20 JANUARY 2003** Morning 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer **all** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 4 printed pages.

[Turn over

1 An insurance company is investigating the amounts of money paid out for each claim on a certain type of insurance policy. It uses the continuous random variable X as a model for the amounts paid out per claim (measured in thousands of pounds), where X has probability density function

$$f(x) = xe^{-x} \qquad \text{for } x \geqslant 0.$$

(i) Use integration by parts to find the cumulative distribution function of X and hence show that, for $t \geqslant 0$,

$$P(X > t) = e^{-t}(1 + t).$$ [5]

(ii) Evaluate $P(X > 2.5)$. [1]

(iii) Verify that the median amount paid out per claim is, to a good approximation, £1680. [2]

(iv) Use the result that $\int_0^{\infty} y^n e^{-y}\,dy = n!$ for $n = 0, 1, 2, \ldots$ to find the mean and variance of X. [4]

(v) A manager decides to investigate the Normal distribution with mean 2 and variance 2 as a model for the amounts (in thousands of pounds) paid out per claim. Find the probability given by this model that an individual pay-out will exceed £2500. [2]

(vi) Fig. 1 is a sketch of the graph of $f(x)$. Make a rough copy of this sketch and draw on the same axes a rough sketch of the probability density function of the $N(2, 2)$ distribution. Indicate clearly the areas that correspond with the probabilities calculated in parts **(ii)** and **(v)**. [2]

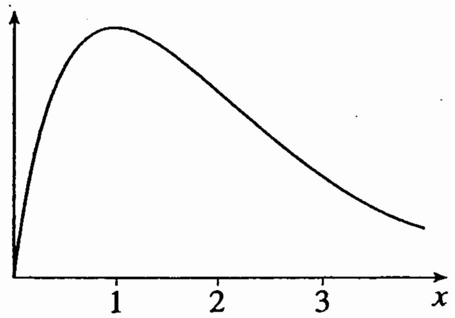

Fig. 1

[Total 16]

2 The senior technician of a university science department is considering using a new supplier of chemicals for student laboratory experiments. These chemicals do not need to be of a very high grade of purity, but must be sufficiently pure that the experiments can be conducted efficiently and safely. The new supplier states that the average level of impurity in the chemicals it supplies is no more than 4%. The senior technician carefully measures the impurity in a random sample of eight containers of these chemicals and finds the percentages to be as follows.

3.9 4.5 4.7 4.3 4.9 3.4 4.5 5.0

(i) State the appropriate null and alternative hypotheses for the usual t test for examining whether the supplier is meeting the stated standard. [2]

(ii) Explain why the corresponding test based on the N(0, 1) distribution cannot be used. [1]

(iii) What condition is necessary for the correct use of the t test? [1]

(iv) Carry out the test, using a 5% significance level. [7]

(v) Provide a two-sided 95% confidence interval for the true mean percentage of impurity in the supplier's chemicals. [4]

[Total 15]

3 Batteries used to power a heavy-duty torch for a particular industrial application have life-lengths, in hours, given by the Normally distributed random variable X with mean 25 and standard deviation 1.2.

The torch contains three of these batteries. These are wired in such a way that the first of them powers the torch until that battery reaches the end of its life. The second battery is then brought into use until it reaches the end of its life. Similarly, the third battery is then brought into use until it reaches the end of its life. The torch is then discarded and replaced by a new one containing three new batteries.

(i) The torch displays a warning message when the third battery is brought into use. Find the probability that the torch will work for at least 28 hours after this message first appears. [2]

(ii) Taking the lives of the batteries as independent, find the probability that a new torch will work for more than 72 hours. [4]

(iii) A trial is conducted with a different design of battery whose life-length is given by the random variable $3X$. An experimental torch is modified so as to contain just one of these batteries. Find the probability that this torch will work for more than 72 hours. [4]

(iv) It is decided to revert to the original design of battery and torch, but a trial is made of batteries from a different manufacturer. Each of a random sample of 20 such batteries has its life-length determined; the sum of the life-lengths is 503.3 hours.

Assuming that the standard deviation of battery life-length from this manufacturer is the same as that for the original batteries, provide a two-sided 90% confidence interval for the mean life-length of batteries from this manufacturer. [4]

[Total 14]

[Turn over

4 The manager of a chain of stationery shops keeps records concerning sales of Christmas cards. The cards are categorised broadly into four types: large traditional cards, small traditional cards, humorous cards and special cards for close relatives. Over the chain as a whole, the proportions sold in these four categories are 29%, 36%, 13% and 22% respectively.

(i) A sample consisting of the first 200 cards purchased one Saturday at one of the shops was taken. They were classified as follows.

Large traditional	Small traditional	Humorous	Special
67	70	13	50

Regarding this as a random sample of the population of all cards purchased at that shop, test at the 5% level of significance whether the proportions may be taken as the same as those for the chain as a whole. [6]

(ii) Give two reasons why the sample in part (i) is unlikely to be representative of its population. [2]

(iii) Inspection of till records for the chain as a whole shows that the amount of money spent on Christmas cards by each customer may be taken as a random variable with mean £6.85 and standard deviation £4.40. Give two reasons why a Normal distribution is unlikely to be a satisfactory model for the amounts spent by individual customers.

Find the probability that the total amount spent by 200 randomly chosen customers will exceed £1500. [7]

[Total 15]

Mark Scheme

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Q1	$f(x) = xe^{-x},\ x \geq 0$, ($x$ in thousands of pounds).			
(i)	C.d.f. $F(t) = \int_0^t xe^{-x}dx$ $= \left[-xe^{-x}\right]_0^t + \int_0^t e^{-x}dx$ $= \left[-te^{-t}\right] - [0] + \left[-e^{-x}\right]_0^t$ $= -te^{-t} - e^{-t} + e^0$ $= 1 - e^{-t} - te^{-t}$ $\therefore P(X > t) = 1 - \textit{this} = e^{-t}(1+t)$	M1 M1 A1 A1 A1	Set up required integral including limits. Reasonable attempt to integrate by parts. Successful integration to $-xe^{-x} - e^{-x}$. Limits used to obtain correct c.d.f. c.a.o. c.a.o. ANSWER GIVEN.	5
(ii)	$P(X > 2.5) = 3.5e^{-2.5} = 0.2873$	B1		1
(iii)	Median of X, m, is given by $\frac{1}{2} = e^{-m}(1+m)$ Inserting $m = 1.68$ in RHS gives $2.68\ e^{-1.68} = 0.4995 \approx 0.5$ as required.	M1 A1	Definition of median. Convincingly shown.	2
(iv)	$E(X) = \int_0^\infty x^2e^{-x}dx$ $= 2!$ (from the given result) $= 2$ $E(X^2) = \int_0^\infty x^3e^{-x}dx = 3! = 6$ $\therefore Var(X) = E(X^2) - \{E(X)\}^2 = 6 - 4 = 2$	M1 A1 M1 A1	Set up required integral with limits. c.a.o. Set up required integral and evidence of intention to use the definition of variance. ft c's $E(X)$ only.	4
(v)	Using N(2, 2), $P(X > 2.5) = P(\ N(0,1) > \frac{2.5-2}{\sqrt{2}} = 0.3535(5))$ $= 1 - 0.6381 = 0.3619$	M1 A1	 Accept use of $z = 0.353$ or 0.354 leading to probabilities in the range 0.3617 to 0.3621.	2
(vi)	Sketch showing $f(x)$ and N(2, 2).	B1 B1	N(2, 2) near enough correct. BOTH areas *clearly* marked.	2
				16

Q2 (i)	$H_0 : \mu = 4$ $H_1 : \mu > 4$ (Do **not** allow $\bar{x} = ...$ or similar.) Where μ is the (population) mean percentage of impurity in the supplier's chemicals.	B1 B1	*Both* must be correct. Allow statements in words (see below). Must indicate "mean"; condone "average". If the symbol used is not μ, or no symbol is used, then insist on "population".	2
(ii)	Sample is small and true σ^2 is unknown.	B1	*Both* statements must be present.	1
(iii)	Underlying distribution is Normal.	B1	Must be describing the population and not the sample or sample means. There may be other supporting evidence e.g. $X \sim N(.. , ..)$ earlier.	1
(iv)	$\bar{x} = 4.4$, $s_{n-1} = 0.5318$ $(s_{n-1}{}^2 = 0.2829)$ Test statistics is $\frac{4.4-4}{\left(\frac{0.5318}{\sqrt{8}}\right)}$ $= 2.12(73)$	B1 M1 A1	Allow $s_n = 0.4975$, $(s_n{}^2 = 0.2475)$ only if correctly used in sequel. Allow alternative: $4 +$ (c's 1.895) $\times \frac{0.5318}{\sqrt{8}}$ (= 4.356) for subsequent comparison with $\bar{x}$. c.a.o. (but ft from here if this is wrong.) Use of $4 - \bar{x}$ (or $\bar{x} - 1.895 \times \frac{0.5318}{\sqrt{8}}$ = 4.044) scores M1A0, but next 4 marks still available (must have lower s.t. point if that mark is to be awarded).	3
	Refer to t_7. Single-tailed 5% point is 1.895. Significant. Seems mean percentage of impurity is not as stated.	M1 A1 A1 A1	No ft from here if wrong. No ft from here if wrong. ft only c's test statistic. ft only c's test statistic. S.C. (t_7 and 2.365) or (t_8 and 1.860) can score max 1 of the last 2 marks if either form of conclusion is given, consistent with the test statistic and critical value.	4
(v)	CI is given by 4.4 ± 2.365 $\times \frac{0.5318}{\sqrt{8}}$ $= 4.4 \pm 0.4447$ $= (3.95(53), 4.84(47))$	M1 B1 M1 A1	c's $\bar{x}$ ± ... c's $s_{n-1}/\sqrt{8}$ c.a.o. Min 2 dp required. Must be given as an interval. N.B. The *same* wrong distribution used can score max M1B0M1A0. ZERO out of 4 if candidate *changes* to a wrong distribution. Recovery to t_7 can score full marks.	4
				15

Q3	$X \sim N(25, \sigma = 1.2\ [\sigma^2 = 1.44])$			
(i)	$P(X \geq 28) = P(N(0, 1) \geq \frac{28-25}{1.2} = 2.5)$	M1	Award *once*, here or elsewhere, for standardising.	
	$= 1 - 0.9938 = 0.0062$	A1	c.a.o.	2
(ii)	$X_1 + X_2 + X_3 \sim N(75,$	B1	Mean.	
	$1.44 + 1.44 + 1.44 = 4.32 = 2.0785^2)$	M1	Variance.	
	$P(X_1 + X_2 + X_3 > 72) = P(N(0, 1) > \frac{72-75}{\sqrt{4.32}} = -1.443)$	M1	Interpret the situation using c's parameters, and leading to a probability greater than 0.5 in the right hand tail.	
	$= 0.9255$	A1	c.a.o.	4
(iii)	$3X \sim N(75,$	B1	Mean.	
	$9 \times 1.2^2 = 12.96 = 3.6^2)$	M1	Variance.	
	$P(3X > 72) = P(N(0, 1) > \frac{72-75}{\sqrt{12.96}} = -0.8333\ \ldots)$	M1	Interpret the situation using c's parameters, and leading to a probability greater than 0.5 in the right hand tail.	
	$= 0.7976$	A1	c.a.o.	4
	OR use:			
	$P(X > □ \times 72$	M1	Alternative solution.	
	$= 24)$	A1		
	$= P(N(0, 1) > \frac{24-25}{1.2} = -0.8333\ \ldots)$	M1	Interpret the situation, and leading to a probability greater than 0.5 in the right hand tail.	
	$= 0.7976$	A1	c.a.o.	
(iv)	$\Sigma x = 503.3,\ \bar{x} = 25.165$			
	90% CI is given by $25.165 \pm$	M1	c's $\bar{x}$ ± …	
	1.645	B1		
	$\times \frac{1.2}{\sqrt{20}}$	M1		
	$= 25.165 \pm 0.4414$ $= (24.72(36), 25.60(64))$	A1	c.a.o. Min 2 dp required. Must be given as an interval. ZERO out of 4 if candidate *changes* to a wrong distribution, such as t_{19} or t_{20}.	4
				14

Q4 (i)	o_i: 67, 70, 13, 50 e_i: 58, 72, 26, 44			
	$X^2 = \frac{(67-58)^2}{58} + \ldots$	M1	Reasonable attempt to calculate X^2.	
	$= 1.3966 + 0.0555 + 6.5 + 0.8182$ $= 8.77(03)$	A1	c.a.o. (but ft from here if this is wrong.)	2
	Refer to χ_3^2.	M1	No ft from here if wrong.	
	Upper 5% point 7.815.	A1	No ft from here if wrong.	
	Significant.	A1	ft only c's test statistic.	
	Seems proportions are not the same.	A1	ft only c's test statistic.	4
	Note – on this occasion no further discussion of results is expected.		S.C. χ_4^2 and 9.488 used can score max 1 of the last 2 marks if either form of conclusion is given, consistent with the test statistic and critical value.	
(ii)	e.g. Saturdays are unlikely to be typical.	E1		
	First 200 are unlikely to be typical.	E1	Allow any two sensible reasons why the sample is unlikely to be representative of *this* population.	2
(iii)	e.g. "Practical" argument that there will be some very large purchases leading to an asymmetric distribution.	E1		
	"Statistical" argument that $N(6.85, 4.40^2)$ has a high probability of giving impossible negative values.	E1	Allow other reasons if sensible.	2
	$X_i \sim$ (mean = 6.85, SD = 4.40)			
	By CLT $Total = X_1 + X_2 + \ldots X_{200} \sim N(1370, 3872)$	B1	Normal, possibly implied.	
		B1	Mean.	
		M1	Variance, or SD = 62.225.	
	$P(Total > 1500) = P(\, N(0, 1) > \frac{1500-1370}{\sqrt{3872}} = 2.089)$	M1	Standardising.	
	$= 1 - 0.9817 = 0.0183$	A1	c.a.o.	5
				15

Examiner's Report

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General Comments

There were 334 candidates from 66 centres (January 2002: 336 from 68). The general standard of the scripts seen was pleasing: many candidates were clearly well prepared for this paper. The improvement in their ability to tackle the routine calculations in the main areas of the syllabus has been maintained. However, as in the past, candidates' explanations often showed that they had not thought carefully enough about what they were being asked to comment on.

Invariably all four questions were attempted. Marks for Question 3 were consistently high: many candidates worked through it quickly and easily, and almost half scored full marks for it. Candidates were least successful in Question 1, particularly in parts (i) and (iv) which required the setting up and evaluation of appropriate integrals.

Comments on Individual Questions

Question 1 (Continuous random variables; insurance payouts)

(i) Many candidates struggled with this part of the question. In fact very few completely correct answers were seen. Candidates did not seem to be well versed in integration by parts and there were many attempts that were unsuccessful because of sign errors. However a more worrying aspect of the answers to this part was the apparent lack of understanding of how to find the cumulative distribution function, especially setting up the relevant integral including limits.

(ii) This part of the question was almost always correct.

(iii) Most candidates managed to write down the equation $e^{-m}(1 + m) = 0.5$ to define the median, and many of them verified that $m = 1.68$ gives a good approximation. But there were also many abortive attempts to solve the equation directly using, for example, logs.

(iv) Many candidates had little difficulty in using the given result to find the mean and variance of the distribution. There were just as many who seemed unable to appreciate the relevance of the result and who made little, if any, progress.

(v) This part of the question was almost always correct.

(vi) Generally this part of the question was answered correctly, but the quality of curve sketching was poor. Often the p.d.f. of N(2, 2) started at the origin and returned to the x-axis at about $x = 4$. Sometimes the shading did not succeed in making the required regions absolutely obvious.

(i) $F(x) = 1 - e^{-x} - xe^{-x}$; (ii) 0.2873; (iii) Mean 2, variance 2; (iv) 0.3619

Question 2 (Hypothesis test and confidence interval for the population mean using the t distribution; impurities in chemicals)

(i) Most candidates stated their hypotheses in an acceptable form. However the vast majority did not define adequately (if at all) the symbol used (μ). We wish candidates to make it clear that they are testing the population mean. This point has been referred to in previous reports.

(ii) Responses to this part were not as good as had been hoped. Many candidates gave only one of the two reasons expected. Many seemed not to appreciate the distinction between this and the next part of the question.

(iii) 'Underlying Normality' in one form or another was often seen, but it was frequently accompanied by other suggestions that were irrelevant or incorrect or were strays from the previous part.

(iv) The test was generally done well. The most common error was the use of the wrong critical value, usually 2.365.

(v) The confidence interval was usually found efficiently and correctly.

(iv) test statistic 2.127, critical value 1.895; (v) (3.955, 4.845)

Question 3 (Combinations of Normal distributions and confidence interval for the population mean; life-lengths of batteries)

(i) This part of the question was almost always correct.

(ii) In this part some difficulties were encountered in identifying the correct variance.

(iii) Again there were difficulties over the variance. Also a handful of candidates managed to get themselves confused over this and the previous part. Having answered this part they then decided that in fact this answer was really the answer to part (ii). So they relabelled their answers accordingly and, in many cases, ended up losing marks as a consequence.

(iv) As in Question 2, the confidence interval was worked out efficiently in the vast majority of cases. A few candidates tried mistakenly to use the t distribution with $v = 19$ or 20.

(i) 0.0062; (ii) 0.9255; (iii) 0.7976; (iv) (24.72, 25.60)

Question 4 (Chi-squared test for goodness of fit; Central Limit Theorem; sales of Christmas cards)

(i) Most candidates conducted the chi-squared test correctly and with ease.

(ii) It seemed that many candidates did not understand the difference between 'representative' and 'random'. It was quite common to read answers which began 'It is not random because ...' or 'It is not representative because it is not random'. Furthermore, few, it seemed, had actually taken the trouble to read the question carefully enough to discover what the population under discussion was, i.e. the cards purchased at the particular shop. So answers that referred to the population of customers or the population of shops had missed the point. There was also much discussion of such things as taste and the time of year.

(iii) The quality of criticism of the Normal distribution as a model for the amounts of money spent was disappointing. Comments about the taste, wealth and buying habits of customers abounded whereas an understanding of the principles of modelling and the characteristics of the Normal distribution was in short supply. In contrast to that, there were many correct calculations of the probability using the Central Limit Theorem. However there were also many candidates who quoted the wrong variance (e.g. $(200 \times 4.40)^2$) and/or used it incorrectly (e.g. as σ^2/n) in the standardisation process. Several candidates made both of these mistakes and so ended up with the correct probability but for the wrong reasons.

(i) test statistic 8.770, critical value 7.815; (iii) 0.0184