OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2605

Pure Mathematics 5

Monday **23 JUNE 2003** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 3 printed pages and 1 blank page.

[Turn over

1 The equation $x^3 + 2x^2 + 8x - 5 = 0$ has roots α, β and γ.

(i) Write down the values of $\alpha + \beta + \gamma$, $\beta\gamma + \gamma\alpha + \alpha\beta$ and $\alpha\beta\gamma$. [3]

(ii) Find $\alpha^2 + \beta^2 + \gamma^2$. [2]

(iii) Show that $(\beta\gamma)^2 + (\gamma\alpha)^2 + (\alpha\beta)^2 = 84$. [3]

(iv) Find $\alpha^2\beta + \alpha^2\gamma + \beta^2\gamma + \beta^2\alpha + \gamma^2\alpha + \gamma^2\beta$. [4]

(v) Find a cubic equation with integer coefficients which has roots

$$2\alpha + \beta\gamma,\ 2\beta + \gamma\alpha \text{ and } 2\gamma + \alpha\beta.$$ [8]

2 (i) Prove that $\operatorname{arsinh} x = \ln(x + \sqrt{x^2+1})$. [5]

(ii) Find $\displaystyle\int_0^2 \frac{1}{\sqrt{4x^2+9}}\,dx$, giving your answer in logarithmic form. [5]

(iii) Given that $f(x) = \operatorname{arsinh}(\frac{4}{3} + x)$, find $f'(x)$ and $f''(x)$. [3]

(iv) Find the Maclaurin series for $\operatorname{arsinh}(\frac{4}{3} + x)$, as far as the term in x^2. [3]

(v) Hence show that, when h is small, $\displaystyle\int_{-h}^{h} x \operatorname{arsinh}(\tfrac{4}{3} + x)\,dx \approx \tfrac{2}{5}h^3$. [4]

3 (i) Express $e^{-\frac{1}{2}j\theta} - e^{\frac{1}{2}j\theta}$ in trigonometric form, and show that $(1 - e^{j\theta})^2 = -4e^{j\theta}\sin^2\frac{1}{2}\theta$. [6]

(ii) For a positive integer n, series C and S are given by

$$C = 1 - \binom{2n}{1}\cos\theta + \binom{2n}{2}\cos 2\theta - \binom{2n}{3}\cos 3\theta + \ldots + \cos 2n\theta,$$

$$S = \quad -\binom{2n}{1}\sin\theta + \binom{2n}{2}\sin 2\theta - \binom{2n}{3}\sin 3\theta + \ldots + \sin 2n\theta.$$

Show that $C = (-4)^n \cos n\theta \sin^{2n}(\frac{1}{2}\theta)$, and find a similar expression for S. [9]

(iii) Given that $w = e^{j\phi}$ is a cube root of 1, state the three possible values of ϕ with $-\pi < \phi < \pi$, and find the possible values of $(1 - w)^6$. [5]

4 (a) A curve has polar equation $r = k\sin 2\theta$, for $0 \leqslant \theta \leqslant 2\pi$, where k is a positive constant.

(i) Sketch the curve, using a continuous line for sections where $r > 0$ and a broken line for sections where $r < 0$. [3]

(ii) The point A has cartesian coordinates $(\frac{\sqrt{3}}{4}k, \frac{3}{4}k)$.

Find the polar coordinates of A, and hence show that A lies on the curve. [3]

(iii) Find the area of one loop of the curve. [5]

(b) A conic has polar equation $\dfrac{3a}{r} = 3 + 2\cos\theta$, where a is a positive constant.

(i) Sketch the conic. [4]

The point P(R, ϕ) lies on the conic, with $0 < \phi < \frac{1}{2}\pi$, and the line PO (where O is the origin) meets the conic again at the point Q.

(ii) Show that $\text{OP} + \text{OQ} = \dfrac{18a}{9 - 4\cos^2\phi}$. [3]

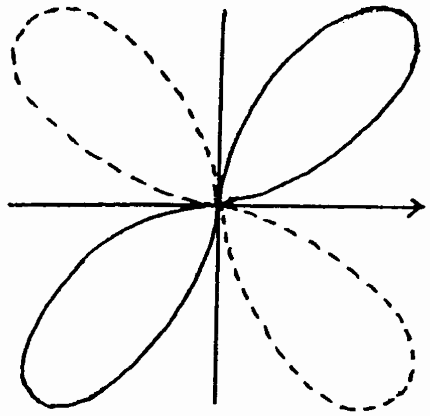
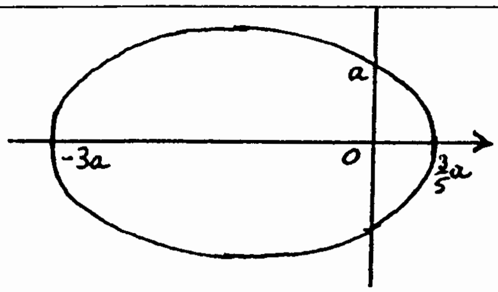
(iii) Given that $\text{PQ} = 3a$, find ϕ. [2]

Mark Scheme

1 (i)	$\sum\alpha=-2,\ \sum\alpha\beta=8,\ \alpha\beta\gamma=5$	B1B1B1 3	
(ii)	$\sum\alpha^2=\left(\sum\alpha\right)^2-2\sum\alpha\beta=(-2)^2-2(8)$ $=-12$	M1 A1 cao 2	Correct formula and substituting numbers
(iii)	$\sum(\alpha\beta)^2=\left(\sum\alpha\beta\right)^2-2\alpha\beta\gamma\sum\alpha$ $=8^2-2(5)(-2)=84$	M1A1 A1 (ag) 3	
(iv)	$\sum\alpha^2\beta=\left(\sum\alpha\right)\left(\sum\alpha\beta\right)-3\alpha\beta\gamma$ $=(-2)(8)-3(5)$ $=-31$	M1A1 M1 A1 cao 4	*Dependent on previous M1*
(v)	$\sum(2\alpha+\beta\gamma)=2\sum\alpha+\sum\alpha\beta$ $=4$ $\sum(2\alpha+\beta\gamma)(2\beta+\gamma\alpha)$ $=4\sum\alpha\beta+2\sum\alpha^2\beta+(\alpha\beta\gamma)\sum\alpha$ $=-40$ $(2\alpha+\beta\gamma)(2\beta+\gamma\alpha)(2\gamma+\alpha\beta)$ $=8(\alpha\beta\gamma)+4\sum(\alpha\beta)^2+2(\alpha\beta\gamma)\sum\alpha^2+(\alpha\beta\gamma)^2$ $=281$ Equation is $y^3-4y^2-40y-281=0$	B1 M1A1 M1A2 M1 A1 cao 8	M1 requires not more than one error M1 requires not more than two errors Give A1 for 3 terms correct *Dependent on at least M1* A0 if ' = 0 ' is missing *SR:* Give B4 for exactly correct answer obtained by approximate (or no) working

2 (i)	$x=\frac{1}{2}(e^y-e^{-y})$ (where $y=\text{arsinh}\,x$) $e^{2y}-2xe^y-1=0$ $e^y=\dfrac{2x\pm\sqrt{4x^2+4}}{2}$ $\left(=x\pm\sqrt{x^2+1}\right)$ $e^y=x+\sqrt{x^2+1}$ (since $e^y>0$) $y=\ln(x+\sqrt{x^2+1})$	M1 M1 M1 A1 A1 (ag)	5	Not dependent on previous A1
	OR If $x=\sinh y$ then $\sqrt{x^2+1}=\cosh y$ M1 $x+\sqrt{x^2+1}=\sinh y+\cosh y$ M1 $=\frac{1}{2}(e^y-e^{-y})+\frac{1}{2}(e^y+e^{-y})=e^y$ M1A1 $\text{arsinh}\,x=y=\ln(x+\sqrt{x^2+1})$ A1			OR Differentiating both sides M1 Method for differentiating $\ln(x+\sqrt{x^2+1})$ M1 Both derivatives correct and shown to be equal A1 Considering constant M1 Showing constant is zero A1
(ii)	$\int_0^2\dfrac{1}{\sqrt{4x^2+9}}\,dx=\left[\dfrac{1}{2}\text{arsinh}\dfrac{2x}{3}\right]_0^2$ $=\frac{1}{2}\text{arsinh}\frac{4}{3}$ $=\frac{1}{2}\ln 3$	M1 A1 A1 M1 A1	5	For arsinh, *or* any sinh substitution For $\text{arsinh}\frac{2}{3}x$ *or* $2x=3\sinh\theta$ For factor $\frac{1}{2}$ *or* $\int\frac{1}{2}d\theta$ Using $\text{arsinh}\,x=\ln(x+\sqrt{x^2+1})$ *SR:* $\text{arsinh}\frac{4}{3}=\ln(4+\sqrt{4^2+3^2})$ M1A0
	OR M2 $\int_0^2\dfrac{1}{\sqrt{4x^2+9}}\,dx=\left[\dfrac{1}{2}\ln\left(2x+\sqrt{4x^2+9}\right)\right]_0^2$ A1A1 $=\frac{1}{2}(\ln 9-\ln 3)=\frac{1}{2}\ln 3$ A1			Integral of form $\ln(ax+\sqrt{b^2x^2+c^2})$ For $\frac{1}{2}$ and $\ln(2x+\sqrt{4x^2+9})$ [*or* $\ln(x+\sqrt{x^2+\frac{9}{4}})$]
(iii)	$f'(x)=\left\{(\frac{4}{3}+x)^2+1\right\}^{-\frac{1}{2}}$ $f''(x)=-(\frac{4}{3}+x)\left\{(\frac{4}{3}+x)^2+1\right\}^{-\frac{3}{2}}$	B1 M1 A1	3	Requires linear factor and power $-\frac{3}{2}$ *SR:* M1A0 for correct differentiation of $\left\{(\frac{4}{3}+x)^2+1\right\}^{\frac{1}{2}}$
(iv)	$f(0)=\ln 3$, $f'(0)=\frac{3}{5}$, $f''(0)=-\frac{36}{125}$ Maclaurin series is $\text{arsinh}(\frac{4}{3}+x)=\ln 3+\frac{3}{5}x-\frac{18}{125}x^2+\dots$	M1 A2 ft	3	Evaluating (at least two) when $x=0$ Give A1 for two terms correct For $\ln 3$ accept $\text{arsinh}\frac{4}{3}$ or 1.09 to 1.1 or ft a wrong value of $\text{arsinh}\frac{4}{3}$ clearly stated earlier
(v)	$I=\int_{-h}^{h}(x\ln 3+\frac{3}{5}x^2-\frac{18}{125}x^3+\dots)\,dx$ $=\left[\frac{1}{2}x^2\ln 3+\frac{1}{5}x^3-\frac{9}{250}x^4+\dots\right]_{-h}^{h}$ $\approx(\frac{1}{2}h^2\ln 3+\frac{1}{5}h^3-\frac{9}{250}h^4)-(\frac{1}{2}h^2\ln 3-\frac{1}{5}h^3-\frac{9}{250}h^4)$ $=\frac{2}{5}h^3$	M1 A1 ft M1 A1 (ag)	4	*In (iv) and (v) coefficients must be non-zero to qualify for ft* Substitution of limits, and cancellation of terms in h^2 and h^4 Allow ft from $a+\frac{3}{5}x+bx^2$ *Not dependent on previous A1*

3 (i)	$e^{-\frac{1}{2}j\theta} - e^{\frac{1}{2}j\theta} = (\cos\frac{1}{2}\theta - j\sin\frac{1}{2}\theta) - (\cos\frac{1}{2}\theta + j\sin\frac{1}{2}\theta)$ $= -2j\sin\frac{1}{2}\theta$	M1 A1	One term correct
	$1 - e^{j\theta} = e^{\frac{1}{2}j\theta}\left(e^{-\frac{1}{2}j\theta} - e^{\frac{1}{2}j\theta}\right) = -2je^{\frac{1}{2}j\theta}\sin\frac{1}{2}\theta$ $(1 - e^{j\theta})^2 = -4e^{j\theta}\sin^2\frac{1}{2}\theta$	M1A1 M1A1 (ag) 6	*Or* $2\sin\frac{1}{2}\theta(\sin\frac{1}{2}\theta - j\cos\frac{1}{2}\theta)$
	OR $(1 - e^{j\theta})^2 = 1 - 2e^{j\theta} + e^{2j\theta}$ $= (1 - 2\cos\theta + \cos 2\theta) + j(-2\sin\theta + \sin 2\theta)$ $= 2\cos^2\theta - 2\cos\theta + j(2\sin\theta\cos\theta - 2\sin\theta)$ M1A1 $= 2(\cos\theta - 1)(\cos\theta + j\sin\theta)$ $= (-4\sin^2\frac{1}{2}\theta)e^{j\theta}$ M1A1		Expressing in terms of $\sin\theta$, $\cos\theta$ and $e^{j\theta}$ only (M1A0 if one error) M1 for using $1 - \cos\theta = 2\sin^2\frac{1}{2}\theta$
(ii)	$C + jS$ $= 1 - \binom{2n}{1}e^{j\theta} + \binom{2n}{2}e^{2j\theta} - \binom{2n}{3}e^{3j\theta} + \ldots + e^{2nj\theta}$ $= (1 - e^{j\theta})^{2n}$ $= \left(-4e^{j\theta}\sin^2\frac{1}{2}\theta\right)^n$ $= (-4)^n e^{jn\theta}\sin^{2n}\frac{1}{2}\theta$ Equating real parts, $C = (-4)^n\cos n\theta\sin^{2n}\frac{1}{2}\theta$ Equating imaginary parts, $S = (-4)^n\sin n\theta\sin^{2n}\frac{1}{2}\theta$	M1 A1 M1A1 M1 A1 M1 A1 (ag) A1 9	Forming $C + jS$ (in any form) M1 for recognising binomial
(iii)	$\phi = 0,\ \pm\frac{2}{3}\pi$ $(1 - w)^6 = (1 - e^{j\phi})^6 = \left(-4e^{j\phi}\sin^2\frac{1}{2}\phi\right)^3$ $= -64e^{3j\phi}\sin^6\frac{1}{2}\phi$ When $\phi = 0$, $(1 - w)^6 = 0$ When $\phi = \pm\frac{2}{3}\pi$, $(1 - w)^6 = -64(1)(\pm\frac{\sqrt{3}}{2})^6 = -27$	B1 M1 A1 B1 A1 5	For general ϕ or with $\phi = (\pm)\frac{2}{3}\pi$ For other methods (or no working) give B2B1 for -27, -27

4(a)(i)		B1 B1 B1 3	For one loop For two further loops Fully correct with continuous and broken lines
(ii)	$r=\sqrt{(\frac{\sqrt{3}}{4}k)^2+(\frac{3}{4}k)^2}=\frac{\sqrt{3}}{2}k$ $\tan\theta=\dfrac{\frac{3}{4}k}{\frac{\sqrt{3}}{4}k}=\sqrt{3}$, so $\theta=\frac{1}{3}\pi$ Polar coordinates are $\left(\frac{\sqrt{3}}{2}k,\ \frac{1}{3}\pi\right)$ $k\sin 2\theta=k\sin(\frac{2}{3}\pi)=k(\frac{\sqrt{3}}{2})=r$ so A lies on the curve	B1 B1 B1 3	Accept $0.866k$ Accept 1.05 Requires an exact and convincing argument using correct coordinates
(iii)	Area of one loop is $\int_0^{\frac{1}{2}\pi}\frac{1}{2}(k\sin 2\theta)^2\,d\theta=\frac{1}{4}k^2\int_0^{\frac{1}{2}\pi}(1-\cos 4\theta)\,d\theta$ $=\frac{1}{4}k^2\left[\theta-\frac{1}{4}\sin 4\theta\right]_0^{\frac{1}{2}\pi}$ $=\dfrac{\pi k^2}{8}$	M1 A1 A1A1 A1 cao 5	Integral of $\sin^2 2\theta$ Correct integral expression (limits required) For $\int\sin^2 2\theta\,d\theta=\frac{1}{2}(\theta-\frac{1}{4}\sin 4\theta)$ Accept $0.393k^2$ *Not dependent on previous two A1's*
(b)(i)		B1 B1 B1 B1 4	One intercept correct *(intercepts can be implied from a table)* Any ellipse (or circle) Ellipse with origin as RH focus Fully correct, intercepts indicated *SR:* If a is missing, B0B1B1B1 can be awarded
(ii)	$OP=\dfrac{3a}{3+2\cos\phi}$, $OQ=\dfrac{3a}{3+2\cos(\phi-\pi)}$ $OP+OQ=\dfrac{3a}{3+2\cos\phi}+\dfrac{3a}{3-2\cos\phi}$ $=\dfrac{3a(3-2\cos\phi)+3a(3+2\cos\phi)}{(3+2\cos\phi)(3-2\cos\phi)}=\dfrac{18a}{9-4\cos^2\phi}$	B1 M1 A1 (ag) 3	One intermediate step required
(iii)	$PQ=3a$ when $18a=3a(9-4\cos^2\phi)$ $\cos\phi=\frac{\sqrt{3}}{2}$ $\phi=\frac{1}{6}\pi$	M1 A1 2	Accept 0.524

Examiner’s Report

2605 Pure Mathematics 5

General Comments

The candidates demonstrated a wide range of ability on this paper, with about 20% scoring 50 marks or more (out of 60) and about 30% scoring less than 30 marks. Almost all candidates answered questions 1 and 2, then about 60% chose question 3 and about 40% chose question 4. Some candidates did appear to have insufficient time to complete their three questions.

Comments on Individual Questions

Q.1 Roots of a cubic equation

This was the best answered question, with half the attempts scoring 18 marks or more (out of 20) and about 30% scoring full marks. Parts (i), (ii) and (iii) were usually answered correctly. In part (iv), those who considered $\sum\alpha\sum\alpha\beta$ were usually successful, but many candidates considered $\sum\alpha\sum\alpha^2$ and experienced more difficulty, as it was then necessary to find $\sum\alpha^3$. The principles involved in finding the new cubic equation in part (v) were well understood, but many algebraic and numerical slips were made; in particular the term $(\alpha\beta\gamma)^2$, which occurs in the product of the new roots, was frequently lost at some stage in the working.

(i) $-2,\ 8,\ 5$; (ii) -12; (iv) -31; (v) $y^3-4y^2-40y-281=0$.

Q.2 Hyperbolic functions

This question was also well answered, with half the attempts scoring 14 marks or more. Part (i) was well understood, although many candidates lost a mark for failing to explain why one of the roots can be rejected. The integral in part (ii) was usually found correctly. In parts (iii) and (iv) the methods for finding a Maclaurin series were well known, but there were many errors, usually in finding the second derivative. In part (v), quite a few candidates neglected the third term of the series, rather than including it and showing that it makes no contribution to the definite integral.

(ii) $\frac{1}{2}\ln 3$; (iii) $\mathrm{f}'(x)=\left\{(\frac{4}{3}+x)^2+1\right\}^{-\frac{1}{2}}$, $\mathrm{f}''(x)=-(\frac{4}{3}+x)\left\{(\frac{4}{3}+x)^2+1\right\}^{-\frac{3}{2}}$;
(iv) $\ln 3+\frac{3}{5}x-\frac{18}{125}x^2+\ldots$.

Q.3 Complex numbers

This was the worst answered question, with half the attempts scoring 5 marks or less. In part (i), the first result was usually correct, but the second result caused great difficulty. Some candidates answered part (ii) quickly and confidently, but the majority either omitted it completely or tried to make it into a geometric series. In part (iii) many candidates earned 2 marks for the values of ϕ and the zero value of $(1-w)^6$, but few saw how the result in part (i) could be used to complete the question.

(i) $-2\mathrm{j}\sin\frac{1}{2}\theta$; (ii) $S=(-4)^n\sin n\theta\sin^{2n}\frac{1}{2}\theta$; (iii) $\phi=0,\ \pm\frac{2}{3}\pi$, $(1-w)^6=0,\ -27$.

Q.4 Polar coordinates

The average mark for this question was about 11. In part (a)(i) the loops were usually correctly drawn, although the continuous and broken lines were not always in the right places. Parts (a)(ii) and (iii) were answered well. In part (b)(i) most candidates drew an ellipse, but the intercepts on the axes were very often wrong; in particular the origin was frequently at the centre of the ellipse. Part (b)(ii) was often omitted, and few candidates answered it successfully. The final part (b)(iii) was often answered correctly.

(a)(ii) $\left(\frac{\sqrt{3}}{2}k,\ \frac{1}{3}\pi\right)$; (iii) $\frac{1}{8}\pi k^2$; (b)(iii) $\frac{1}{6}\pi$.