OCR
RECOGNISING ACHIEVEMENT

Oxford, Cambridge and RSA Examinations

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2602/1

Pure Mathematics 2

Thursday **11 JANUARY 2001** Morning 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.

Answer **all** questions.

You are permitted to use only a scientific calculator for this paper.

INFORMATION FOR CANDIDATES

The approximate allocation of marks is given in brackets [ ] at the end of each question or part question.

You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.

Final answers should be given to a degree of accuracy appropriate to the context.

The total number of marks for this paper is 60.

This question paper consists of 4 printed pages.

HN/3

1 Fig. 1 shows part of the graph with equation $y = x\sqrt{9-2x^2}$. It crosses the x-axis at $(a, 0)$.

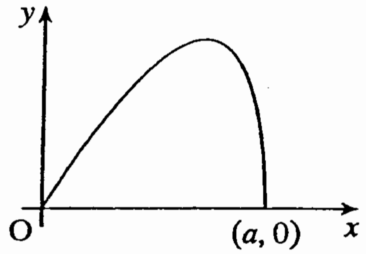

Fig. 1

(i) Find the value of a, giving your answer as a multiple of $\sqrt{2}$. [3]

(ii) Show that the result of differentiating $\sqrt{9-2x^2}$ is $\dfrac{-2x}{\sqrt{9-2x^2}}$.

Hence show that if $y = x\sqrt{9-2x^2}$ then

$$\frac{dy}{dx} = \frac{9-4x^2}{\sqrt{9-2x^2}}.$$ [7]

(iii) Find the x-coordinate of the maximum point on the graph of $y = x\sqrt{9-2x^2}$.

Write down the gradient of the curve at the origin. What can you say about the gradient at the point $(a, 0)$? [5]

2 (a) The sum of the first n terms of the arithmetic progression 50, 52, 54, 56, ... is denoted by S.

The sum of the first n terms of the arithmetic progression 100, 99, 98, 97, ... is denoted by T.

(i) Express each of S and T in terms of n. [3]

(ii) Deduce the least value of n for which $S > T$. [4]

(b) The sequence u_n is defined by

$$u_n = n \sin(a + 180n)°, \quad n = 1, 2, 3, 4, \ldots$$

where a is a constant, and $0 < a < 90$.

(i) Write down and simplify the first 4 terms of the sequence. Find the sum of these 4 terms, giving your answer in terms of $\sin a°$. [5]

(ii) Deduce the value of $\sum_{n=1}^{100} u_n$, giving your answer in terms of $\sin a°$. [3]

3 Fig. 3 shows the graph of $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \frac{x}{1+x^2}.$$

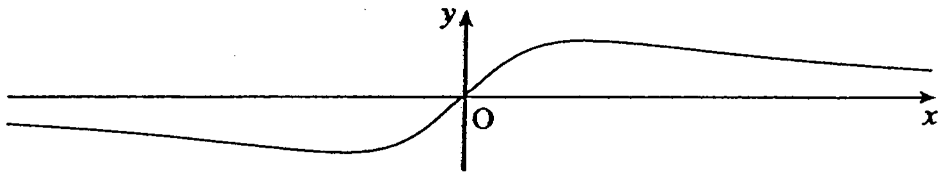

Fig. 3

(i) Show algebraically that $\mathrm{f}(x)$ is an odd function. State what feature of the graph corresponds to the fact that $\mathrm{f}(x)$ is an odd function. [2]

(ii) Find, using calculus, the coordinates of the stationary points on the graph of $y = \mathrm{f}(x)$. Verify that the maximum and minimum points are as shown in Fig. 3.

Justify the shape of the graph as $x \to \infty$ and as $x \to -\infty$. [9]

(iii) Find the area of the finite region between the graph of $y = \mathrm{f}(x)$, the x-axis and the line $x = 1$, giving your answer in terms of a logarithm. [4]

[Turn over

4 **(a)** The function f(x) is defined by

$$f(x) = (1.1)^x,$$

so that, for example, $f(2) = (1.1)^2 = 1.21$.

(i) Write down the value of $f(0)$. [1]

(ii) Given that

$$f(a) = \frac{1}{(1.1)^2}, \qquad f(b) = \sqrt[3]{1.1} \qquad \text{and} \qquad f(c) = 1.5,$$

write down the values of a and b, and calculate the value of c correct to 3 decimal places. [5]

(b) Fig. 4 shows the graph of $y = (1.1)^x$ for values of x between 0 and 10, with ten rectangles of width 1 drawn below the curve. Thus, for example, the height of the third rectangle is $(1.1)^2$, as marked in Fig. 4. The total area of the ten rectangles is denoted by A.

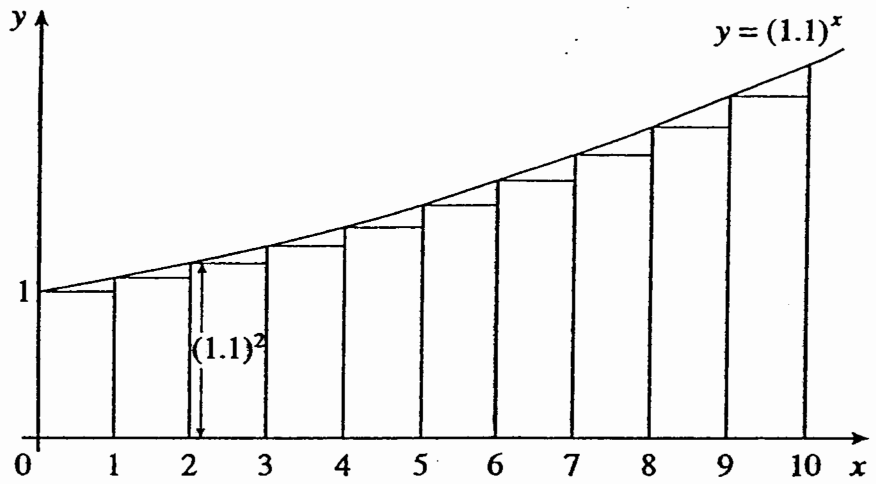

Fig. 4

(i) Show that the areas of these rectangles form a geometric progression, and deduce that

$$A = 10\left[(1.1)^{10} - 1\right].$$

Evaluate A correct to 2 decimal places. [4]

(ii) It can be shown that, for all values of x,

$$(1.1)^x = e^{kx},$$

where k is a constant equal to $\ln(1.1)$.

Using this result, show that the exact area under the curve $y = (1.1)^x$ between $x = 0$ and $x = 10$ is $\dfrac{A}{10\ln(1.1)}$. Evaluate this expression correct to 2 decimal places. [5]

Mark Scheme

1(i) At a, $x\sqrt{9-2x^2})=0$ $\Rightarrow 2x^2=9$ $\Rightarrow x^2=\frac{9}{2} \Rightarrow x=\frac{3}{\sqrt{2}}$ $=\frac{3\sqrt{2}}{2}$	M1 A1 A1 cao [3]	soi Any correct expression for a $\frac{3\sqrt{2}}{2}$ or $1.5\sqrt{2}$ www
(ii) $\frac{d}{dx}(\sqrt{9-2x^2})=\frac{1}{2}(9-2x^2)^{-\frac{1}{2}}.(-4x)$ $=-\frac{2x}{\sqrt{9-2x^2}}$ * $y=x\sqrt{9-2x^2}$ $\Rightarrow \frac{dy}{dx}=1.\sqrt{9-2x^2}-\frac{2x^2}{\sqrt{9-2x^2}}$ $=\frac{9-2x^2-2x^2}{\sqrt{9-2x^2}}$ $=\frac{9-4x^2}{\sqrt{9-2x^2}}$ *	B1 B1 E1 B1 B1 M1 E1 [7]	$\frac{1}{2}u^{-\frac{1}{2}}$ or $\frac{1}{2}(9-2x^2)^{-\frac{1}{2}}$.(-4x) (condone missing brackets) www $1.\sqrt{9-2x^2}$ $-\frac{2x^2}{\sqrt{9-2x^2}}$ expressing their two terms as a single fraction www – must show intermediate step
(iii) Maximum point when $9-4x^2=0$ $\Rightarrow x^2=\frac{9}{4}$ $\Rightarrow x=\frac{3}{2}$ Gradient at origin $=\frac{9}{\sqrt{9}}$ $=3$ At $(a,0)$, gradient is infinite	M1 A1 M1 A1 B1 [5]	soi or 1.5 substituting $x=0$ into $\frac{dy}{dx}$ (soi) cao $\pm\infty$ or equivalent statement Accept 'undefined'

2 (a) (i) $S = \frac{n}{2}(100 + [n-1]2)$ $= \frac{n}{2}(98 + 2n)$ or $n(49 + n)$ $T = \frac{n}{2}[200 + (n-1)(-1)]$ $= \frac{n}{2}[201 - n]$	B2 B1 [3]	Any correct expression in terms of n for sum of S or T (need not be simplified) Any correct expression in terms of n for sum of the other series
(ii) $S > T$ $\Rightarrow 98 + 2n > 201 - n$ $\Rightarrow 3n > 103$ $\Rightarrow n > 34\frac{1}{3}$ so after 35 terms.	M1 M1 A1 A1ft [4]	Forming an (in)equality for n Collecting terms for (their) (in)equality $n > 34\frac{1}{3}$ or $n = 34\frac{1}{3}$ soi n = next whole number above their $34\frac{1}{3}$ (but must be positive)
(b) (i) $u_1 = 1.\sin(a + 180) = -\sin a$ $u_2 = 2.\sin(a + 360) = 2 \sin a$ $u_3 = 3.\sin(a + 540) = -3 \sin a$ $u_4 = 4.\sin(a + 720) = 4 \sin a$ $\Rightarrow u_1+u_2+u_3+u_4 = -\sin a + 2 \sin a - 3 \sin a + 4 \sin a$ $= 2 \sin a$	M1 B3 A1cao [5]	Substituting n = 1, 2, 3 and 4 into $n \sin(a + 180n)$ Correct expressions for u_1, u_2, u_3 and u_4 in terms of $\sin a$. -1 for each error, www. Accept $u_1 = \sin(-a)$, $u_3 = 3 \sin(-a)$
(ii) $\sum_{n=1}^{100} u_n = -\sin a + 2 \sin a - 3 \sin a + 4 \sin a$ $-5\sin a + 6 \sin a - 7 \sin a + 8 \sin a$ ... $-97\sin a + 98\sin a - 99\sin a + 100\sin a$ $= 25 \times 2 \sin a$ $= 50 \sin a$	M1 M1 A1cao [3]	Interpreting the summation notation Organising terms to allow summation (or using AP formula to sum their AP)

3 (i) $f(-x) = \frac{-x}{1+(-x)^2} = -\frac{x}{1+x^2} = -f(x)$	B1	Must show $(-x)^2$.
Half turn symmetry about O	B1 [2]	Condone no centre of rotation, but must imply rotation and half turn or 180° or order 2
(ii) $f'(x) = \frac{(1+x^2).1 - x.2x}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$	B1 B1	Using quotient rule: correct numerator correct denominator
or $f'(x) = 1.(1+x^2)^{-1} + x.(-1)(1+x^2)^{-2}.2x$ $= (1+x^2)^{-2}(1+x^2-2x^2) = \frac{1-x^2}{(1+x^2)^2}$	B1 B1	Using product rule: $1.(1+x^2)^{-1}$ $x.(-1)(1+x^2)^{-2}.2x$
$f'(x) = 0$ when $1 - x^2 = 0$ $\Rightarrow x = 1$ or -1.	M1 A1ft	sets their numerator to zero $x = 1, -1$ or $(1, \frac{1}{2})$
When $x = 1, y = \frac{1}{2}$, when $x = -1, y = -\frac{1}{2}$	A1ft	$y = \frac{1}{2}, -\frac{1}{2}$ or $(-1, -\frac{1}{2})$ (ft on equivalent work)
x: -1^-, -1, -1^+, 1^-, 1, 1^+ $\frac{dy}{dx}$: $-$, 0, $+$, $+$, 0, $-$		
or $\frac{d^2y}{dx^2} = \frac{(1+x^2)^2.(-2x) - (1-x^2).2(1+x^2).2x}{(1+x^2)^4}$ $= \frac{-2x(3x^2-1)}{(1+x^2)^3}$	M1	determining nature of one TP using 2nd derivative test or by testing sign of gradient either side.
When $x = 1$, $\frac{d^2y}{dx^2} = -\frac{1}{2} < 0 \Rightarrow$ maximum When $x = -1$, $\frac{d^2y}{dx^2} = \frac{1}{2} > 0 \Rightarrow$ minimum	A1 [7]	www (e.g. correct 2nd derivative and substitutions for both TPs
$f(x) = \frac{1}{\frac{1}{x} + x} \to 0^+$ as $x \to \infty$ and 0^- as $x \to -\infty$	B1 B1 [2]	States $f(x) \to 0$ or $\frac{1}{x}$ [at $\pm\infty$] Justifies this - allow a verification for a large value of x. Allow arguments using '∞'.
(iii) Area $= \int_0^1 \frac{x}{1+x^2}dx$ let $u = 1 + x^2$	M1	Correct integral and limits (condone no dx)
$du = 2xdx$ $= \int_1^2 \frac{1}{2u}du$	M1	$\int \frac{1}{2u}du$ (condone wrong limits)
$= \frac{1}{2}[\ln u]_1^2 = \frac{1}{2}\ln 2$	A1 A1 [4]	$\frac{1}{2}\ln u$ or B2 $\frac{1}{2}\ln(1+x^2)$ cao

4 (a) (i) f(0) = 1	B1 [1]	
(ii) $a = -2$ $b = \frac{1}{3}$ $1.1^c = 1.5$ $\Rightarrow c \ln 1.1 = \ln 1.5$ $\Rightarrow c = \frac{\ln 1.5}{\ln 1.1} = 4.254$ to 3 d.p.	B1 B1 M1 M1 A1 cao [5]	or −2.000 or 0.333 (not 0.33) Allow SCB1 for 1.1^{-2} and $1.1^{1/3}$ seen use of logs or trial and improvement must show some method
(b) (i) Areas are 1, 1.1, 1.1^2, …This is a GP with 1st term 1 and common ratio 1.1 $A = 1 + 1.1 + 1.1^2 + \ldots + 1.1^9$ $= \frac{1.1^{10} - 1}{1.1 - 1}$ $= 10(1.1^{10} - 1)$ * = 15.94 to 2 d.p.	B1 M1 E1. B1 cao [4]	$r = 1.1$ or convincing statement (e.g. area is multiplied by 1.1 from 1 rectangle to the next) or $\frac{1-1.1^{10}}{1-1.1}$ www
(ii) Exact area = $\int_0^{10} 1.1^x \, dx$ $= \int_0^{10} e^{kx} dx$ $= \left[\frac{1}{k}e^{kx}\right]_0^{10}$ $= \frac{1}{k}(e^{10k} - 1)$ $= \frac{1}{k}(1.1^{10} - 1)$ $= \frac{1}{\ln 1.1} \cdot \frac{A}{10}$ $= \frac{A}{10 \ln 1.1}$ * = 16.72	 M1 A1 B1 E1 B1 [5]	$\int e^{kx} dx$ or $\int e^{x \ln 1.1} dx$ (condone wrong limits or no dx) $\frac{1}{k}e^{kx}$ or $\frac{1}{\ln 1.1}e^{x \ln 1.1}$ $= \frac{1}{k}(e^{10k} - 1)$ or $\frac{1}{\ln 1.1}(1.1^{10} - 1)$ [$k = \ln 1.1$] www 16.68 – 16.72, must be to 2 d.p.

Examiner’s Report

Pure Mathematics 2 (2602)

General Comments

This report was written to cover both 2602 and 5502 candidates, as the paper was common. The entry for 2602 was small and consisted largely of able candidates, for whom the prevalent errors reported below may not be entirely relevant.

There was a very variable standard of work on the paper, with scores across the full range of marks. While there were many excellent scripts with marks over 50, there were equally some very poor ones, with scores in single figures, from candidates who clearly had not mastered the basic techniques required by the syllabus, and were ill prepared to tackle the paper. The distribution of marks appeared to be somewhat bimodal, with many scores in the 20s and 40s, but fewer in the 30s.

The main cause of concern, as previous reports of this paper have commented upon, is the standard of algebra. Many candidates appear to understand the principles of calculus, series work and functions, but their solutions are let down by the brittleness of their algebraic manipulation skills. Sloppy notation such as negative expressions without brackets, and integrals without dx, was equally commonplace. For this paper, students are expected to handle equations with polynomial fractions, and it would be pleasing to lay to rest the misconception, commonly held in both questions 1 and 3, that $\frac{f(x)}{g(x)} = 0 \Rightarrow f(x) = g(x)$.

Most candidates appeared to complete the paper without time problems, although calculating second derivatives in Question 1 and in Question 3 (rather than testing the sign of the gradient), lost a lot of time for few marks gained. Trigonometric transformations, as tested in question 2(*b*), were poorly understood.

Comments on Individual Questions

Question 1 (Graph and gradient)

Most candidates scored reasonably well on this question. In part (i), virtually all candidate achieved the method mark for equating the function to zero, but only the better candidates got to $\frac{3}{2}\sqrt{2}$, most stopping at $\frac{3}{\sqrt{2}}$ or its equivalent, so manipulation of surds does not appear to be well known. More disturbing was the popularity of $\sqrt{9-2x^2} = 3 - \sqrt{2}x$. In part (ii), the chain rule was well done, and most candidates applied the product rule to produce $\sqrt{9-2x^2} - \frac{2x^2}{\sqrt{9-2x^2}}$ but failed to derive, or fudged, the given answer. In part (iii), many lost time through checking the nature of the maximum, which was not asked for by the question. $\frac{\sqrt{9-4x^2}}{\sqrt{9-2x^2}} = 0 \Rightarrow \sqrt{9-4x^2} = \sqrt{9-2x^2}$ was not an uncommon error. The gradient at the origin was generally found, but that at $(a, 0)$ was less well done, $\frac{-9}{0} = 0$ being seen too often.

(i) $\frac{3\sqrt{2}}{2}$, (iii) 1.5, 3, 'infinite' or equivalent (e.g. 'undefined')

Question 2 (Series and sequence)

Part (*a*) was often fully correct. Common errors were to compare the terms rather than the sums of terms, omitting the negative sign in the formula for T, and poor algebraic simplification to solve the inequality. Candidates who checked $n = 34$ and $n = 35$ correctly gained full marks.

Part (*b*), on the other hand, was universally poorly answered. Very few candidates seemed to know what to do with $\sin(a + 180)$, etc., and burned their boats with $\sin(a+180) = \sin a + \sin 180$. Some follow through was allowed in the second part for summing $\sin a$, $\sin 2a$, etc. to end up with $5050\sin a$.

(*a*) (i) $S = n(49+n)$, $T = \frac{n}{2}[201-n]$ (ii) $n = 35$
(*b*) (i) $-\sin a$, $2\sin a$, $-3\sin a$, $4\sin a$; $2\sin a$ (ii) $50\sin a$.

Question 3 (Function, stationary point, area)

Candidates had quite a lot to do for the marks. In part (i), there were plenty of good derivations of $f(-x) = -f(x)$. Some thought that $f(-x) \neq f(x)$ sufficed to define an odd function; $\frac{-x}{1+-x^2} = -\frac{x}{1+x^2}$ was not condoned. In the geometric interpretation, they needed to give the order to rotational symmetry to gain the mark. In part (ii), the quotient rule was usually successful as far as $\frac{1+x^2-2x^2}{(1+x^2)^2}$, but a disturbing number of candidates failed to simplify this to $\frac{1-x^2}{(1+x^2)^2}$. Equally worrying was the number of candidates who then set both numerator *and* denominator to zero to find the turning point, producing some spurious solutions for x in the process. Many candidates spent inordinate amounts of time struggling with the second derivative, rather than checking the sign of the gradient either side of the stationary values. When checking the limit of the function as $x \to \pm\infty$, candidates needed to mention the limiting value zero (or, better, $1/x$) to gain the mark, but a numerical substitution of a large value of x to verify the limit was condoned. The integration was often well done, although omitting the '$\frac{1}{2}$' from $\frac{1}{2}\ln(1+x^2)$ was common, and the notation for performing an integration by substitution was often poorly expressed.

(ii) $f'(x) = \dfrac{1-x^2}{(1+x^2)^2}$ (1,1/2) maximum , (−1, −1/2) minimum.

(iii) $\frac{1}{2}\ln 2$.

Question 4 (Exponential function, approximation to area)

There were some easy marks around for the weaker candidates, but only the very best got 15 marks. Part (i) was virtually 100% correct, and many candidates got full marks for (ii). Absence of working (trial and improvement or logarithms) sometimes proved costly in finding c. Most gave the answers to the required accuracy.

In (b), some evidence to convince us that the series was a G.P. was required, but many derived the expression for A correctly, and even more managed to evaluate it. Deriving the exact area was much less common, although evaluating this gained a consolation mark for many candidates.

(i) 1 (ii) $a = -2$, $b = \frac{1}{3}$, $c =$ 4.254 (iii) 15.94 (iv) 16.72 .