

1 (a) Evaluate

$$\sum_{r=1}^n \frac{6}{r(r+1)(r+3)}.$$

(b) Expand $\ln(1+x+x^2+x^3)$ as a series in powers of x , where $|x| < 1$, giving the first five non-zero terms and the general term.

(c) Expand $e^{x \ln(1+x)}$ as a series in powers of x , where $-1 < x \leq 1$, as far as the term in x^4 .

2 The distinct points P_1, P_2, P_3, Q_1, Q_2 and Q_3 in the Argand diagram are represented by the complex numbers z_1, z_2, z_3, w_1, w_2 and w_3 respectively. Show that the triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ are similar, with P_i corresponding to Q_i ($i = 1, 2, 3$) and the rotation from 1 to 2 to 3 being in the same sense for both triangles, if and only if

$$\frac{z_1 - z_2}{z_1 - z_3} = \frac{w_1 - w_2}{w_1 - w_3}.$$

Verify that this condition may be written

$$\det \begin{pmatrix} z_1 & z_2 & z_3 \\ w_1 & w_2 & w_3 \\ 1 & 1 & 1 \end{pmatrix} = 0.$$

(i) Show that if $w_i = z_i^2$ ($i = 1, 2, 3$) then triangle $P_1P_2P_3$ is not similar to triangle $Q_1Q_2Q_3$.

(ii) Show that if $w_i = z_i^3$ ($i = 1, 2, 3$) then triangle $P_1P_2P_3$ is similar to triangle $Q_1Q_2Q_3$ if and only if the centroid of triangle $P_1P_2P_3$ is the origin. [The *centroid* of triangle $P_1P_2P_3$ is represented by the complex number $\frac{1}{3}(z_1 + z_2 + z_3)$.]

(iii) Show that the triangle $P_1P_2P_3$ is equilateral if and only if

$$z_2z_3 + z_3z_1 + z_1z_2 = z_1^2 + z_2^2 + z_3^2.$$

3 The function f is defined for $x < 2$ by

$$f(x) = 2|x^2 - x| + |x^2 - 1| - 2|x^2 + x|.$$

Find the maximum and minimum points and the points of inflection of the graph of f and sketch this graph. Is f continuous everywhere? Is f differentiable everywhere?

Find the inverse of the function f , i.e. expressions for $f^{-1}(x)$, defined in the various appropriate intervals.

4 The point P moves on a straight line in three-dimensional space. The position of P is observed from the points $O_1(0, 0, 0)$ and $O_2(8a, 0, 0)$. At times $t = t_1$ and $t = t'_1$, the lines of sight from O_1 are along the lines $\frac{x}{2} = \frac{z}{3}$, $y = 0$ and $x = 0$, $\frac{y}{3} = \frac{z}{4}$ respectively. At times $t = t_2$ and $t = t'_2$, the lines of sight from O_2 are $\frac{x - 8a}{-3} = \frac{y}{1} = \frac{z}{3}$ and $\frac{x - 8a}{-4} = \frac{y}{2} = \frac{z}{5}$ respectively. Find an equation or equations for the path of P .

5 The curve C has the differential equation in polar coordinates

$$\frac{d^2 r}{d\theta^2} + 4r = 5 \sin 3\theta, \quad \text{for } \frac{\pi}{5} \leq \theta \leq \frac{3\pi}{5},$$

and, when $\theta = \frac{\pi}{2}$, $r = 1$ and $\frac{dr}{d\theta} = -2$. Show that C forms a closed loop and that the area of the region enclosed by C is

$$\frac{\pi}{5} + \frac{25}{48} \left[\sin\left(\frac{\pi}{5}\right) - \sin\left(\frac{2\pi}{5}\right) \right].$$

6 The transformation T from $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} x' \\ y' \end{pmatrix}$ in two-dimensional space is given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where u is a positive real constant. Show that the curve with equation $x^2 - y^2 = 1$ is transformed into itself. Find the equations of two straight lines through the origin which transform into themselves.

A line, not necessarily through the origin, which has gradient $\tanh v$ transforms under T into a line with gradient $\tanh v'$. Show that $v' = v + u$.

The lines ℓ_1 and ℓ_2 with gradients $\tanh v_1$ and $\tanh v_2$ transform under T into lines with gradients $\tanh v'_1$ and $\tanh v'_2$ respectively. Find the relation satisfied by v_1 and v_2 that is the necessary and sufficient for ℓ_1 and ℓ_2 to intersect at the same angle as their transforms.

In the case when ℓ_1 and ℓ_2 meet at the origin, illustrate in a diagram the relation between ℓ_1 , ℓ_2 and their transforms.

7 (a) Prove that

$$\int_0^{\pi/2} \ln \sin x \, dx = \int_0^{\pi/2} \ln \cos x \, dx = \frac{1}{2} \int_0^{\pi/2} \ln \sin 2x \, dx - \frac{1}{4} \pi \ln 2.$$

and

$$\int_0^{\pi/2} \ln \sin 2x \, dx = \frac{1}{2} \int_0^{\pi} \ln \sin x \, dx = \int_0^{\pi/2} \ln \sin x \, dx.$$

Hence, or otherwise, evaluate $\int_0^{\pi/2} \ln \sin x \, dx$.

[You may assume that all the integrals converge.]

(b) Given that $\ln u < u$ for $u \geq 1$ deduce that

$$\frac{1}{2} \ln x < \sqrt{x} \quad \text{for} \quad x \geq 1.$$

Deduce that $\frac{\ln x}{x} \rightarrow 0$ as $x \rightarrow \infty$ and that $x \ln x \rightarrow 0$ as $x \rightarrow 0$ through positive values.

(c) Using the results of parts (a) and (b), or otherwise, evaluate $\int_0^{\pi/2} x \cot x \, dx$.

8 (a) The integral I_k is defined by

$$I_k = \int_0^{\theta} \cos^k x \cos kx \, dx.$$

Prove that $2kI_k = kI_{k-1} + \cos^k \theta \sin k\theta$.

(b) Prove that

$$1 + m \cos 2\theta + \binom{m}{2} \cos 4\theta + \cdots + \binom{m}{r} \cos 2r\theta + \cdots + \cos 2m\theta = 2^m \cos^m \theta \cos m\theta.$$

(c) Using the results of parts (a) and (b), show that

$$\begin{aligned} m \frac{\sin 2\theta}{2} + \binom{m}{2} \frac{\sin 4\theta}{4} + \cdots + \binom{m}{r} \frac{\sin 2r\theta}{2r} + \cdots + \frac{\sin 2m\theta}{2m} \\ = \cos \theta \sin \theta + \cos^2 \theta \sin 2\theta + \cdots + \frac{1}{r} 2^{r-1} \cos^r \theta \sin r\theta + \cdots + \frac{1}{m} 2^{m-1} \cos^m \theta \sin m\theta. \end{aligned}$$

9 The parametric equations E_1 and E_2 define the same ellipse, in terms of the parameters θ_1 and θ_2 , (though not referred to the same coordinate axes).

$$\begin{aligned} E_1 : \quad x &= a \cos \theta_1, & y &= b \sin \theta_1, \\ E_2 : \quad x &= \frac{k \cos \theta_2}{1 + e \cos \theta_2}, & y &= \frac{k \sin \theta_2}{1 + e \cos \theta_2}, \end{aligned}$$

where $0 < b < a$, $0 < e < 1$ and $0 < k$. Find the position of the axes for E_2 relative to the axes for E_1 and show that $k = a(1 - e^2)$ and $b^2 = a^2(1 - e^2)$.

[The standard polar equation of an ellipse is $r = \frac{\ell}{1 + e \cos \theta}$.]

By considering expressions for the length of the perimeter of the ellipse, or otherwise, prove that

$$\int_0^\pi \sqrt{(1 - e^2 \cos^2 \theta)} d\theta = \int_0^\pi \frac{1 - e^2}{(1 + e \cos \theta)^2} \sqrt{(1 + e^2 + 2e \cos \theta)} d\theta.$$

Given that e is so small that e^6 may be neglected, show that the value of either integral is

$$\frac{1}{64} \pi (64 - 16e^2 - 3e^4).$$

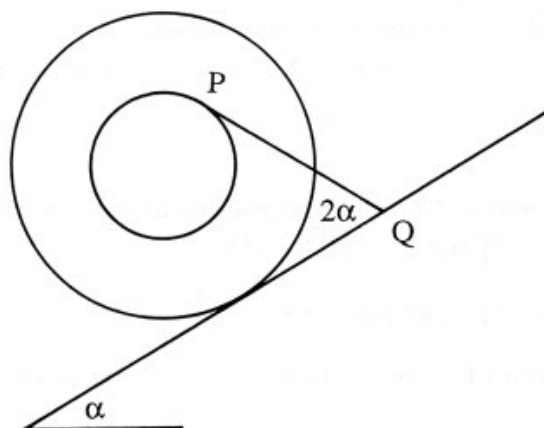
10 The equation $x^n - qx^{n-1} + r = 0$, where $n \geq 5$ and q and r are real constants, has roots $\alpha_1, \alpha_2, \dots, \alpha_n$. The sum of the products of m distinct roots is denoted by Σ_m (so that, for example, $\Sigma_3 = \sum \alpha_i \alpha_j \alpha_k$ where the sum runs over all values of i, j and k with $n \geq i > j > k \geq 1$). The sum of the m th powers of the roots is denoted by S_m (so that, for example, $S_3 = \sum_{i=1}^n \alpha_i^3$). Prove that $S_p = q^p$ for $1 \leq p \leq n - 1$.

[You may assume that for any n th degree equation and $1 \leq p \leq n$

$$S_p - S_{p-1} \Sigma_1 + S_{p-2} \Sigma_2 - \dots + (-1)^{p-1} S_1 \Sigma_{p-1} + (-1)^p p \Sigma_p = 0.]$$

Find expressions for S_n, S_{n+1} and S_{n+2} in terms of q, r and n . Suggest an expression for S_{n+m} , where $m < n$, and prove its validity by induction.

11



A uniform circular cylinder of radius $2a$ with a groove of radius a cut in its central cross-section has mass M . It rests, as shown in the diagram, on a rough plane inclined at an acute angle α to the horizontal. It is supported by a light inextensible string wound round the groove and attached to the cylinder at one end. The other end of the string is attached to the plane at Q , the free part of the string, PQ , making an angle 2α with the inclined plane. The coefficient of friction at the contact between the cylinder and the plane is μ . Show that $\mu \geq \frac{1}{3} \tan \alpha$.

The string PQ is now detached from the plane and the end Q is fastened to a particle of mass $3M$ which is placed on the plane, the position of the string remaining unchanged. Given that $\tan \alpha = \frac{1}{2}$ and that the system remains in equilibrium, find the least value of the coefficient of friction between the particle and the plane.

12 A smooth tube whose axis is horizontal has an elliptic cross-section in the form of the curve with parametric equations

$$x = a \cos \theta \quad y = b \sin \theta$$

where the x -axis is horizontal and the y -axis is vertically upwards. A particle moves freely under gravity on the inside of the tube in the plane of this cross-section. By first finding \ddot{x} and \ddot{y} , or otherwise, show that the acceleration along the inward normal at the point with parameter θ is

$$\frac{ab \dot{\theta}^2}{\sqrt{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)}}.$$

The particle is projected along the surface in the vertical cross-section plane, with speed $2\sqrt{bg}$, from the lowest point. Given that $2a = 3b$, show that it will leave the surface at the point with parameter θ where

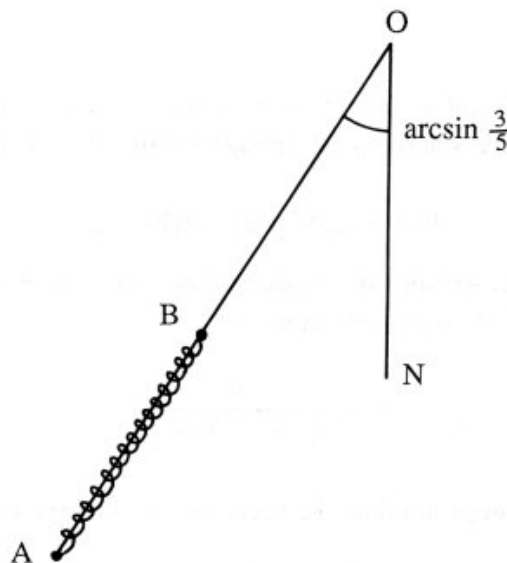
$$5 \sin^3 \theta + 12 \sin \theta - 8 = 0.$$

13 A smooth particle P_1 is projected from a point O on the horizontal floor of a room which has a horizontal ceiling at a height h above the floor. The speed of projection is $\sqrt{(8gh)}$ and the direction of projection makes an acute angle α with the horizontal. The particle strikes the ceiling and rebounds, the impact being perfectly elastic. Show that for this to happen α must be at least $\frac{1}{6}\pi$ and that the range on the floor is then

$$8h \cos \alpha (2 \sin \alpha - \sqrt{(4 \sin^2 \alpha - 1)}).$$

Another particle P_2 is projected from O with the same velocity as P_1 but its impact with the ceiling is perfectly inelastic. Find the difference D between the ranges of P_1 and P_2 on the floor and show that, as α varies, D has a maximum value when $\alpha = \frac{1}{4}\pi$.

14



The end O of a smooth light rod OA of length $2a$ is a fixed point. The rod OA makes a fixed angle $\sin^{-1}(\frac{3}{5})$ with the downward vertical ON , but is free to rotate about ON . A particle of mass m is attached to the rod at A and a small ring B of mass m is free to slide on the rod but is joined to A by a spring of natural length a and modulus of elasticity kmg . The vertical plane containing the rod OA rotates about ON with constant angular velocity $\sqrt{5g/(2a)}$ and B is at rest relative to the rod. Show that the length of OB is

$$\frac{(10k + 8)a}{10k - 9}.$$

Given that the reaction of the rod on the particle at A makes an angle $\tan^{-1}(\frac{13}{21})$ with the horizontal, find the value of k . Find also the magnitude of the reaction between the rod and the ring B .

15 A pack of $2n$ (where $n \geq 4$) cards consists of two each of n different sorts. If four cards are drawn from the pack without replacement show that the probability that no pairs of identical cards have been drawn is

$$\frac{4(n-2)(n-3)}{(2n-1)(2n-3)}$$

Find the probability that exactly one pair of identical cards is included in the four.

If k cards are drawn without replacement and $2 < k < 2n$, find an expression for the probability that there are exactly r pairs of identical cards included when $r < \frac{1}{2}k$.

For even values of k show that the probability that the drawn cards consist of $\frac{1}{2}k$ pairs is

$$\frac{1 \times 3 \times 5 \times \dots \times (k-1)}{(2n-1)(2n-3)\dots(2n-k+1)}$$

16 The random variables X and Y take integer values x and y respectively which are restricted by $x \geq 1$, $y \geq 1$ and $2x + y \leq 2a$ where a is an integer greater than 1. The joint probability is given by

$$P(X = x, Y = y) = c(2x + y),$$

where c is a positive constant, within this region and zero elsewhere. Obtain, in terms of x , c and a , the marginal probability $P(X = x)$ and show that

$$c = \frac{6}{a(a-1)(8a+5)}.$$

Show that when y is an even number the marginal probability $P(Y = y)$ is

$$\frac{3(2a-y)(2a+2+y)}{2a(a-1)(8a+5)}$$

and find the corresponding expression when y is odd.

Evaluate $E(Y)$ in terms of a .