Cambridge Assessment Admissions Testing

Sixth Term Examination Papers **9465**

MATHEMATICS 1 Morning

Tuesday 12 June 2018 Time: 3 hours

5201456514

Additional Materials: Answer Booklet
Formulae Booklet

INSTRUCTIONS TO CANDIDATES

Read this page carefully, but do not open this question paper until you are told that you may do so.

Read the additional instructions on the front of the answer booklet.

Write your name, centre number, candidate number, date of birth, and circle the paper number in the spaces provided on the answer booklet.

Make sure you fill in page 1 **AND** page 3 of the answer booklet with your details.

INFORMATION FOR CANDIDATES

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

You are provided with a Mathematical Formulae Booklet.

Calculators are not permitted.

Wait to be told you may begin before turning this page.

This question paper consists of 8 printed pages and 4 blank pages.

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Section A: Pure Mathematics

1 The line $y = a^2x$ and the curve $y = x(b - x)^2$, where $0 < a < b$, intersect at the origin O and at points P and Q. The x-coordinate of P is less than the x-coordinate of Q. Find the coordinates of P and Q, and sketch the line and the curve on the same axes.

Show that the equation of the tangent to the curve at P is

$$y = a(3a - 2b)x + 2a(b - a)^2.$$

This tangent meets the y-axis at R. The area of the region between the curve and the line segment OP is denoted by S. Show that

$$S = \frac{1}{12}(b - a)^3(3a + b)\,.$$

The area of triangle OPR is denoted by T. Show that $S > \frac{1}{3}T$.

2 If $x = \log_b c$, express c in terms of b and x and prove that $\dfrac{\log_a c}{\log_a b} = \log_b c$.

(i) Given that $\pi^2 < 10$, prove that

$$\frac{1}{\log_2 \pi} + \frac{1}{\log_5 \pi} > 2\,.$$

(ii) Given that $\log_2 \dfrac{\pi}{\mathrm{e}} > \dfrac{1}{5}$ and that $\mathrm{e}^2 < 8$, prove that $\ln \pi > \frac{17}{15}$.

(iii) Given that $\mathrm{e}^3 > 20$, $\pi^2 < 10$ and $\log_{10} 2 > \frac{3}{10}$, prove that $\ln \pi < \frac{15}{13}$.

3 The points R and S have coordinates $(-a\,, 0)$ and $(2a\,, 0)$, respectively, where $a > 0$. The point P has coordinates $(x\,, y)$ where $y > 0$ and $x < 2a$. Let $\angle PRS = \alpha$ and $\angle PSR = \beta$.

(i) Show that, if $\beta = 2\alpha$, then P lies on the curve $y^2 = 3(x^2 - a^2)$.

(ii) Find the possible relationships between α and β when $0 < \alpha < \pi$ and P lies on the curve $y^2 = 3(x^2 - a^2)$.

4 The function f is defined by

$$\mathrm{f}(x) = \frac{1}{x \ln x}\left(1 - (\ln x)^2\right)^2 \qquad (x > 0, \quad x \neq 1)\,.$$

Show that, when $(\ln x)^2 = 1$, both $\mathrm{f}(x) = 0$ and $\mathrm{f}'(x) = 0$.

The function F is defined by

$$\mathrm{F}(x) = \begin{cases} \displaystyle\int_{1/\mathrm{e}}^{x} \mathrm{f}(t)\ \mathrm{d}t & \text{for } 0 < x < 1\,, \\[2ex] \displaystyle\int_{\mathrm{e}}^{x} \mathrm{f}(t)\ \mathrm{d}t & \text{for } x > 1\,. \end{cases}$$

(i) Find $\mathrm{F}(x)$ explicitly and hence show that $\mathrm{F}(x^{-1}) = \mathrm{F}(x)$.

(ii) Sketch the curve with equation $y = \mathrm{F}(x)$.

5 **(i)** Write down the most general polynomial of degree 4 that leaves a remainder of 1 when divided by any of $x - 1$, $x - 2$, $x - 3$ or $x - 4$.

(ii) The polynomial $\mathrm{P}(x)$ has degree N, where $N \geqslant 1$, and satisfies

$$\mathrm{P}(1) = \mathrm{P}(2) = \cdots = \mathrm{P}(N) = 1\,.$$

Show that $\mathrm{P}(N+1) \neq 1$.

Given that $\mathrm{P}(N+1) = 2$, find $\mathrm{P}(N+r)$ where r is a positive integer. Find a positive integer r, independent of N, such that $\mathrm{P}(N+r) = N+r$.

(iii) The polynomial $\mathrm{S}(x)$ has degree 4. It has integer coefficients and the coefficient of x^4 is 1. It satisfies

$$\mathrm{S}(a) = \mathrm{S}(b) = \mathrm{S}(c) = \mathrm{S}(d) = 2001\,,$$

where a, b, c and d are distinct (not necessarily positive) integers.

(a) Show that there is no integer e such that $\mathrm{S}(e) = 2018$.

(b) Find the number of ways the (distinct) integers a, b, c and d can be chosen such that $\mathrm{S}(0) = 2017$ and $a < b < c < d$.

6 Use the identity

$$2\sin P\,\sin Q = \cos(Q-P) - \cos(Q+P)$$

to show that

$$2\sin\theta\left(\sin\theta + \sin 3\theta + \cdots + \sin(2n-1)\theta\right) = 1 - \cos 2n\theta\,.$$

(i) Let A_n be the approximation to the area under the curve $y = \sin x$ from $x = 0$ to $x = \pi$, using n rectangular strips each of width $\frac{\pi}{n}$, such that the midpoint of the top of each strip lies on the curve. Show that

$$A_n \sin\left(\frac{\pi}{2n}\right) = \frac{\pi}{n}\,.$$

(ii) Let B_n be the approximation to the area under the curve $y = \sin x$ from $x = 0$ to $x = \pi$, using the trapezium rule with n strips each of width $\frac{\pi}{n}$. Show that

$$B_n \sin\left(\frac{\pi}{2n}\right) = \frac{\pi}{n}\cos\left(\frac{\pi}{2n}\right).$$

(iii) Show that

$$\frac{1}{2}(A_n + B_n) = B_{2n}\,,$$

and that

$$A_n B_{2n} = A_{2n}^2\,.$$

7 **(i)** In the cubic equation $x^3 - 3pqx + pq(p+q) = 0$, where p and q are distinct real numbers, use the substitution

$$x = \frac{pz+q}{z+1}$$

to show that the equation reduces to $az^3 + b = 0$, where a and b are to be expressed in terms of p and q.

(ii) Show further that the equation $x^3 - 3cx + d = 0$, where c and d are non-zero real numbers, can be written in the form $x^3 - 3pqx + pq(p+q) = 0$, where p and q are distinct real numbers, provided $d^2 > 4c^3$.

(iii) Find the real root of the cubic equation $x^3 + 6x - 2 = 0$.

(iv) Find the roots of the equation $x^3 - 3p^2x + 2p^3 = 0$, and hence show how the equation $x^3 - 3cx + d = 0$ can be solved in the case $d^2 = 4c^3$.

8 The functions s and c satisfy $\mathrm{s}(0) = 0$, $\mathrm{c}(0) = 1$ and

$$\mathrm{s}'(x) = \mathrm{c}(x)^2,$$

$$\mathrm{c}'(x) = -\mathrm{s}(x)^2.$$

You may assume that s and c are uniquely defined by these conditions.

(i) Show that $\mathrm{s}(x)^3 + \mathrm{c}(x)^3$ is constant, and deduce that $\mathrm{s}(x)^3 + \mathrm{c}(x)^3 = 1$.

(ii) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\mathrm{s}(x)\mathrm{c}(x)\Big) = 2\mathrm{c}(x)^3 - 1$$

and find (and simplify) an expression in terms of $\mathrm{c}(x)$ for $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{s}(x)}{\mathrm{c}(x)}\right)$.

(iii) Find the integrals

$$\int \mathrm{s}(x)^2\,\mathrm{d}x \qquad \text{and} \qquad \int \mathrm{s}(x)^5\,\mathrm{d}x\,.$$

(iv) Given that s has an inverse function, s^{-1}, use the substitution $u = \mathrm{s}(x)$ to show that

$$\int \frac{1}{(1-u^3)^{\frac{2}{3}}}\,\mathrm{d}u = \mathrm{s}^{-1}(u) + \text{constant.}$$

(v) Find, in terms of u, the integrals

$$\int \frac{1}{(1-u^3)^{\frac{4}{3}}}\,\mathrm{d}u \qquad \text{and} \qquad \int (1-u^3)^{\frac{1}{3}}\,\mathrm{d}u\,.$$

Section B: Mechanics

9 A straight road leading to my house consists of two sections. The first section is inclined downwards at a constant angle α to the horizontal and ends in traffic lights; the second section is inclined upwards at an angle β to the horizontal and ends at my house. The distance between the traffic lights and my house is d.

I have a go-kart which I start from rest, pointing downhill, a distance x from the traffic lights on the downward-sloping section. The go-kart is not powered in any way, all resistance forces are negligible, and there is no sudden change of speed as I pass the traffic lights. Given that I reach my house, show that $x \sin \alpha \geqslant d \sin \beta$.

Let T be the total time taken to reach my house. Show that

$$\left(\frac{g \sin \alpha}{2}\right)^{\frac{1}{2}} T = (1+k)\sqrt{x} - \sqrt{k^2 x - kd}\ ,$$

where $k = \dfrac{\sin \alpha}{\sin \beta}$.

Hence determine, in terms of d and k, the value of x which minimises T. [You need not justify the fact that the stationary value is a minimum.]

10 A train is made up of two engines, each of mass M, and n carriages, each of mass m. One of the engines is at the front of the train, and the other is coupled between the kth and $(k+1)$th carriages. When the train is accelerating along a straight, horizontal track, the resistance to the motion of each carriage is R and the driving force on each engine is D, where $2D > nR$. The tension in the coupling between the engine at the front and the first carriage is T.

(i) Show that

$$T = \frac{n(mD + MR)}{nm + 2M}\,.$$

(ii) Show that T is greater than the tension in any other coupling provided that $k > \frac{1}{2}n$.

(iii) Show also that, if $k > \frac{1}{2}n$, then at least one of the couplings is in compression (that is, there is a negative tension in the coupling).

11 The point O lies on a rough plane that is inclined at an angle α to the horizontal, where $\alpha < 45°$. The point A lies on the plane a distance d from O up the line L of greatest slope through O. The point B, which is not on the rough plane, lies in the same vertical plane as O and A, and AB is horizontal. The distance from O to B is d.

A particle P of mass m rests on L between O and A. One end of a light inelastic string is attached to P. The string passes over a smooth light pulley fixed at B and its other end is attached to a freely hanging particle of mass λm.

(i) Show that the acute angle, θ, between the string and the line L satisfies $\alpha \leqslant \theta \leqslant 2\alpha$.

(ii) Given that P can rest in equilibrium at every point on L between O and A, show that $2\lambda \sin\alpha \leqslant 1$.

(iii) The coefficient of friction between P and the plane is μ, and the acute angle β is given by $\mu = \tan\beta$. Show that if $\beta \geqslant 2\alpha$, then a necessary condition for equilibrium to be possible for every position of P on L between O and A is

$$\lambda \leqslant \frac{\sin(\beta - \alpha)}{\cos(\beta - 2\alpha)}\,.$$

Obtain the corresponding result if $\alpha \leqslant \beta \leqslant 2\alpha$.

Section C: Probability and Statistics

12 A bag contains three coins. The probabilities of their showing heads when tossed are p_1, p_2 and p_3.

(i) A coin is taken at random from the bag and tossed. What is the probability that it shows a head?

(ii) A coin is taken at random from the bag (containing three coins) and tossed; the coin is returned to the bag and again a coin is taken at random from the bag and tossed. Let N_1 be the random variable whose value is the number of heads shown on the two tosses. Find the expectation of N_1 in terms of p, where $p = \frac{1}{3}(p_1 + p_2 + p_3)$, and show that $\text{Var}(N_1) = 2p(1-p)$.

(iii) Two of the coins are taken at random from the bag (containing three coins) and tossed. Let N_2 be the random variable whose value is the number of heads showing on the two coins. Find $\text{E}(N_2)$ and $\text{Var}(N_2)$.

(iv) Show that $\text{Var}(N_2) \leqslant \text{Var}(N_1)$, with equality if and only if $p_1 = p_2 = p_3$.

13 A multiple-choice test consists of five questions. For each question, n answers are given ($n \geqslant 2$) only one of which is correct and candidates either attempt the question by choosing one of the n given answers or do not attempt it.

For each question attempted, candidates receive two marks for the correct answer and lose one mark for an incorrect answer. No marks are gained or lost for questions that are not attempted. The pass mark is five.

Candidates A, B and C don't understand any of the questions so, for any question which they attempt, they each choose one of the n given answers at random, independently of their choices for any other question.

(i) Candidate A chooses in advance to attempt exactly k of the five questions, where $k = 0, 1, 2, 3, 4$ or 5. Show that, in order to have the greatest probability of passing the test, she should choose $k = 4$.

(ii) Candidate B chooses at random the number of questions he will attempt, the six possibilities being equally likely. Given that Candidate B passed the test find, in terms of n, the probability that he attempted exactly four questions.

(iii) For each of the five questions Candidate C decides whether to attempt the question by tossing a biased coin. The coin has a probability of $\frac{n}{n+1}$ of showing a head, and she attempts the question if it shows a head. Find the probability, in terms of n, that Candidate C passes the test.

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