THE COLLEGES OF OXFORD UNIVERSITY

MATHEMATICS

SUNDAY, 13 DECEMBER, 1998

Time allowed: $2\frac{1}{2}$ hours

For candidates applying for Mathematics, Computation, Mathematics & Philosophy, Mathematics & Computation

Write your name and college (where you are sitting the test) in BLOCK CAPITALS.

NAME:

COLLEGE:

Attempt all the questions. Each of the nine parts of question 1 is worth 4 marks, and questions 2,3,4,5 are worth 16 marks each, giving a total of 100.

Answer question 1 on the grid overleaf. Write your answers to questions 2,3,4,5 in the space provided, continuing on the blank pages at the back of this booklet if necessary.

THE USE OF CALCULATORS OR FORMULA SHEETS IS NOT ALLOWED.

Use this page for rough work if you wish.

1. For each part on pages 4 and 5, exactly one of the answers (i),(ii),(iii),(iv),(v) is correct. Indicate the correct answer with a tick (√) in the corresponding row and column below.

	(i)	(ii)	(iii)	(iv)	(v)
(a)					
(b)					
(c)					
(d)					
(e)					
(f)					
(g)					
(h)					
(k)					

(a) The point lying between $P\,(2,3)$ and $Q\,(8,-3)$ which divides the line PQ in the ratio 1:2 has coordinates
(i) $(4,-1)$, (ii) $(6,-2)$, (iii) $(\frac{14}{3},2)$, (iv) $(4,1)$, (v) none of these.

(b) The polynomial $(x^2-1)(x^2+1)$ is divisible by (that is, has as a factor)
(i) $(x+1)^2$,
(ii) x^3-x^2+x-1,
(iii) x^3+x^2-x+1,
(iv) x^3-1,
(v) none of these.

(c) As the positive integer n becomes very large, $\dfrac{2n^3+3n^2+1}{5n^3+4}$ approaches
(i) $\frac{1}{4}$, (ii) $\frac{2}{5}$, (iii) 1, (iv) ∞, (v) none of these.

(d) The simultaneous equations

$$2x+ay=b,$$
$$x+3y=4,$$

(i) when $a=6$ and b is any value, do not have a solution,
(ii) when $b\neq 8$ and a is any value, have a solution,
(iii) have a solution for any values of a and b,
(iv) when either $a\neq 6$ or $b=8$, have a solution,
(v) satisfy none of these.

(e) The area under the curve $y=1+x^2+\sin 3\pi x$ and above the x-axis between $x=0$ and $x=1$ is
(i) $\frac{1}{3}(4+\frac{2}{\pi})$, (ii) $\frac{1}{3}(4+\frac{1}{\pi})$, (iii) $\frac{4}{3}$, (iv) $\frac{1}{3}(4-\frac{2}{\pi})$, (v) none of these.

(f) The function $y = x^2 e^{5x}$ satisfies the equation

(i) $\ln y = 5x \ln(x^2)$,
(ii) $\frac{dy}{dx} - 5y = e^{5x}$,
(iii) $y^2 = x^4 e^{10x^2}$,
(iv) $\ln(\frac{dy}{dx}) = 5x + \ln x + \ln(5x+2)$,
(v) none of these.

(g) The number of solutions to the equation $\sin\frac{x}{2} + \cos x = 1$ in the range $0 \leqslant x \leqslant 2\pi$ is

(i) 0, (ii) 1, (iii) 2, (iv) 3, (v) none of these.

(h) The graph of the function $y = f(x)$ has a maximum at (1,2) and a minimum at $(5, -5)$. It follows that the graph of the function $y = -2f(3+x)$ has

(i) a maximum at $(-2, -4)$ and a minimum at $(2, 10)$,
(ii) a maximum at $(2, -4)$ and a minimum at $(-2, 10)$,
(iii) a minimum at $(-2, -4)$ and a maximum at $(2, 10)$,
(iv) a minimum at $(2, -4)$ and a maximum at $(-2, 10)$,
(v) none of these.

(k) The graph of $y = |x^2 + 2x - 1|$ lies below the graph of $y = 1$ precisely when

(i) $-1 - \sqrt{3} < x < 1 + \sqrt{3}$,
(ii) $-1 - \sqrt{3} < x < -2$ or $0 < x < \sqrt{3} - 1$,
(iii) $-1 - \sqrt{3} < x < -2$ or $0 < x < 1 + \sqrt{3}$,
(iv) $-2 < x < 0$,
(v) none of these.

Turn over

2. (a) You are given that

$$\frac{1}{(x-1)(x-2)} = \frac{A}{x-2} + \frac{B}{x-1},$$

where A and B are constants. Find the values of A and B.

(b) Simplify

$$\frac{1}{(x-1)^{n+1}(x-2)} - \frac{1}{(x-1)^n(x-2)}.$$

(c) You are given that

$$\frac{1}{(x-1)^n(x-2)} = \frac{A_0}{x-2} + \sum_{i=1}^{n} \frac{A_i}{(x-1)^i},$$

where $A_0, A_1, A_2, \ldots$ are constants. Using your answers to (a) and (b), or otherwise, find the values of these constants.

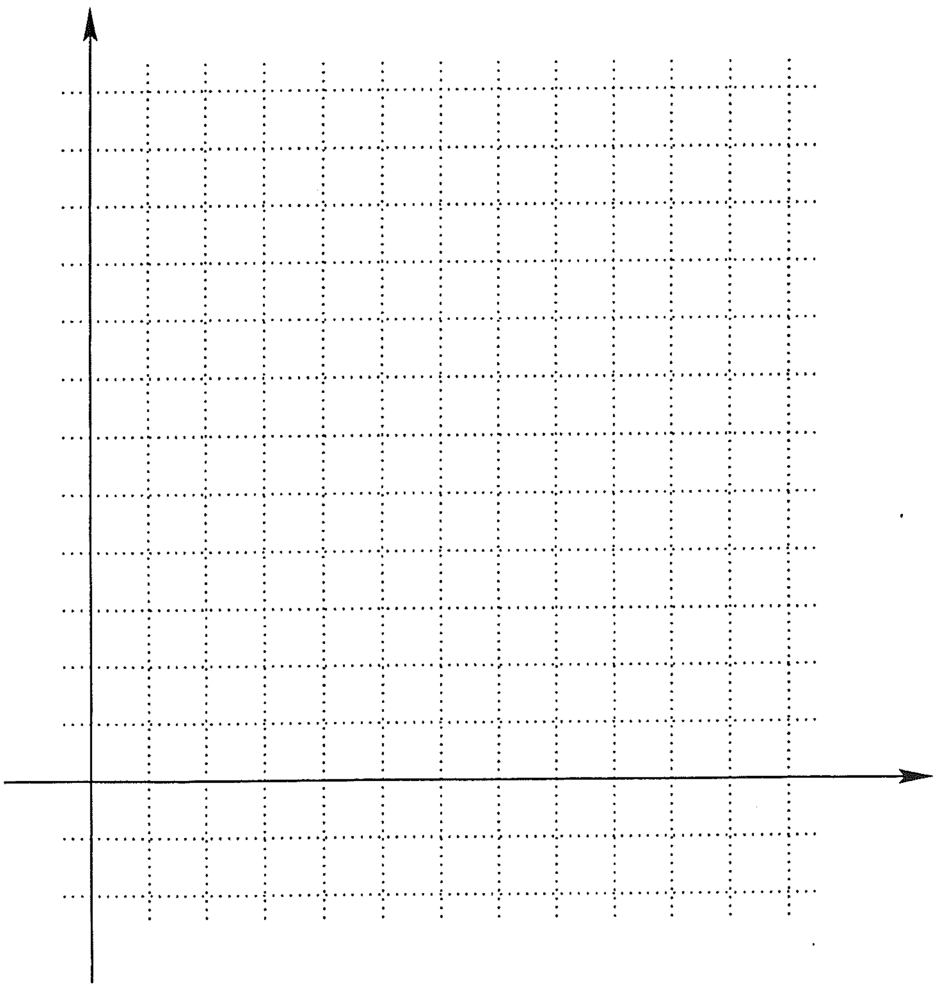
3. Let P and Q be the points with co-ordinates (7, 1) and (11, 2).

(a) The *mirror image* of the point P in the x-axis is the point R with co-ordinates (7, −1). Mark the points P, Q and R on the grid provided.

(b) Consider paths from P to Q each of which consists of two straight line segments PX and XQ where X is a point on the x-axis. Find the length of the shortest such path, giving clear reasoning for your answer. (You may refer to the diagram to help your explanation, if you wish.)

(c) Sketch in the line ℓ with equation $y = x$. Find the co-ordinates of S, the mirror image in the line ℓ of the point Q, and mark in the point S.

(d) Consider paths from P to Q each of which consists of three straight line segments PY, YZ and ZQ, where Y is on the x-axis and Z is on the line ℓ. Find the length of the shortest such path, giving clear reasoning for your answer.



4. (a) Find $\frac{dy}{dx}$ for each of the functions

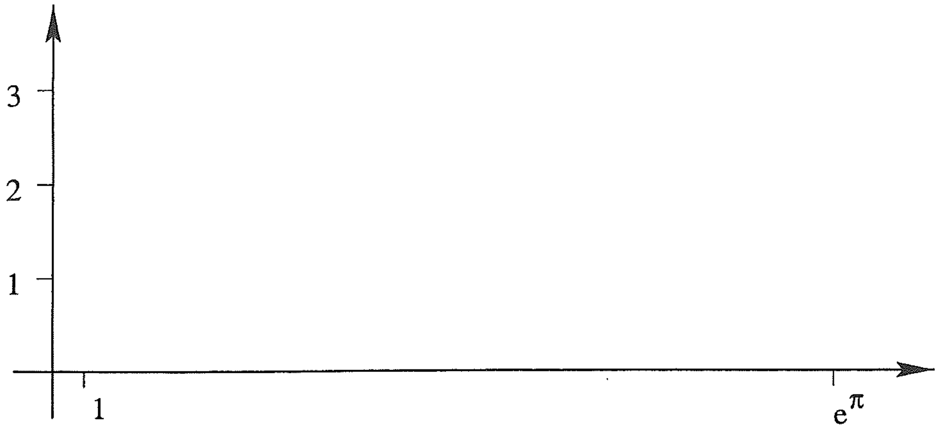
$y = \sin(\ln x)$,
$y = x\sin(\ln x)$,
$y = x\cos(\ln x)$.

(b) Sketch the following curves using the axes provided on the next page:

(i) $y = \ln x, \quad \text{for } 1 \leqslant x \leqslant e^{\pi}$,
(ii) $y = \sin(\ln x), \quad \text{for } 1 \leqslant x \leqslant e^{\pi}$.

(c) Evaluate

$$\int_{1}^{e^{\pi}} \sin(\ln x)dx.$$



Turn over

5. With an unlimited supply of black pebbles and white pebbles, there are 4 ways in which you can put two of them in a row: BB, BW, WB and WW.

(a) Write down the 8 different ways in which you can put three of the pebbles in a row. In how many different ways can you put N of the pebbles in a row?

Suppose now that you are not allowed to put black pebbles next to each other: with two pebbles there are now only 3 ways of putting them in a row, because BB is forbidden.

(b) Write down the 5 different ways that are still allowed for three pebbles. Now let r_N be the number of possible arrangements for N pebbles in a row, still under the no-two-black-together restriction, so that $r_2 = 3$ and $r_3 = 5$.

(c) Show that for $N \geqslant 4$ we have $r_N = r_{N-1} + r_{N-2}$. [Hint: Consider separately the two possible cases for the colour of the last pebble.]

Finally, suppose that we impose the further restriction that the first pebble and the last pebble cannot both be black. For N pebbles call the number of such arrangements w_N, so that for example $w_3 = 4$ (although $r_3 = 5$, the arrangement BWB is now forbidden).

(d) When $N \geqslant 5$, write down a formula for w_N in terms of the numbers r_i, and explain why it is correct.