United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 1 : Friday, 2 December 2011

Time allowed $3\frac{1}{2}$ hours.

Instructions

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.*
- *Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by your solutions in question number order.*
- *Staple all the pages neatly together in the top* left *hand corner.*
- *To accommodate candidates sitting in other time-zones, please do not discuss the paper on the internet until 8am GMT on Saturday 3 December.*

Do not turn over until **told to do so.**

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1. Find all (positive or negative) integers n for which $n^2 + 20n + 11$ is a perfect square. *Remember that you must justify that you have found them all.*

2. Consider the numbers $1, 2, \ldots, n$. Find, in terms of n, the largest integer t such that these numbers can be arranged in a row so that all consecutive terms differ by at least t.

3. Consider a circle S. The point P lies outside S and a line is drawn through P, cutting S at distinct points X and Y. Circles S_1 and S_2 are drawn through P which are tangent to S at X and Y respectively. Prove that the difference of the radii of S_1 and S_2 is independent of the positions of P, X and Y.

4. Initially there are m balls in one bag, and n in the other, where $m, n > 0$. Two different operations are allowed:

   a) Remove an equal number of balls from each bag;
   b) Double the number of balls in one bag.

   Is it always possible to empty both bags after a finite sequence of operations?

   Operation b) is now replaced with

   b′) Triple the number of balls in one bag.

   Is it now always possible to empty both bags after a finite sequence of operations?

5. Prove that the product of four consecutive positive integers cannot be equal to the product of two consecutive positive integers.

6. Let ABC be an acute-angled triangle. The feet of the altitudes from A, B and C are D, E and F respectively. Prove that $DE + DF \leq BC$ and determine the triangles for which equality holds.

   The altitude from A is the line through A which is perpendicular to BC. The foot of this altitude is the point D where it meets BC. The other altitudes are similarly defined.