Quadratic Sequences

Last lesson we learnt how to find an expression for the nth term of a arithmetic (linear) sequence; i.e. one that goes up or down by a constant amount. We found

$$T = (\text{diff between terms})n + (\text{zeroth term}).$$

So for example the sequence $3, -1, -5, -9\ldots$ gives the formula $T = -4n+7$. We spot arithmetic sequences by looking at the first difference and spotting the constant.

However, if the second difference is constant then you are dealing with a *quadratic sequence*. To work out the formula;

1. Halve the second difference to get the number of n^2s (e.g. if you had a second difference of $+6$ you would have $+3n^2$).
2. Write out the original sequence above the terms of your number of n^2s.
3. Subtract the n^2s from the sequence to give the RESIDUE.
4. The residue will either be constant or a linear sequence. If it is a linear sequence then work out its formula.
5. Finally add the number of n^2s to the formula for the residue and this will be the formula for the original sequence.

An Example

Find the formula for the sequence $3, 13, 27, 45, 67\ldots$
Difference table:

$$\begin{array}{ccccccccc} 3 & & 13 & & 27 & & 45 & & 67 \\ & +10 & & +14 & & +18 & & +22 & \\ & & +4 & & +4 & & +4 & & \end{array}$$

So we have quadratic sequence with $+2n^2$ (half of $+4$).
So write out sequence and $+2n^2$ and subtract to find residue.

SEQUENCE	3	13	27	45	67
$2n^2$	2	8	18	32	50
RESIDUE	1	5	9	13	17

So the formula for the residue is $4n-3$ so the overall formula for the sequence $3, 13, 27, 45, 67\ldots$ is $T = 2n^2 + 4n - 3$.

Questions

1. Find the formula for the nth term of the following sequences:
   (a) $3, \quad 6, \quad 11, \quad 18, \quad 27, \quad 38, \quad 51\ldots$ $\boxed{n^2+2}$
   (b) $-19, \quad -15, \quad -9, \quad -1, \quad 9, \quad 21, \quad 35\ldots$ $\boxed{n^2+n-21}$
   (c) $4, \quad 10, \quad 20, \quad 34, \quad 52, \quad 74, \quad 100\ldots$ $\boxed{2n^2+2}$
   (d) $2, \quad 9, \quad 22, \quad 41, \quad 66, \quad 97, \quad 134\ldots$ $\boxed{3n^2-2n+1}$
   (e) $2, \quad 12, \quad 26, \quad 44, \quad 66, \quad 92, \quad 122\ldots$ $\boxed{2n^2+4n-4}$

(f) 14, 67, 122, 179, 238, 299, 362 . . . $n^2 + 50n - 37$

2. Use the formulae for the above expressions to work out
   (a) the 50th term of the sequence,
   (b) the 1000th term of the sequence.

3. Now work out an expression for the nth term of the following mixture of arithmetic and quadratic sequences:
   (a) 3, 3.5, 4, 4.5, 5, 5.5, 6 . . . $T = 0.5n + 2.5$
   (b) 3, 9, 17, 27, 39, 53, 69 . . . $T = n^2 + 3n - 1$
   (c) 13, 6, −1, −8, −15, −22, −29 . . . $T = -7n + 20$
   (d) −2, 4, 14, 28, 46, 68, 94 . . . $T = 2n^2 - 4$
   (e) −8.5, −10, −11.5, −13, −14.5, −16, −17.5 . . . $T = -1.5n - 7$
   (f) 1, 5, 15, 31, 53, 81, 115, . . . $T = 3n^2 - 5n + 3$
   (g) 21.1, 22.2, 23.3, 24.4, 25.5, 26.6, 27.7 . . . $T = 1.1n + 20$
   (h) −0.6, 2.8, 6.2, 9.6, 13, 16.4, 19.8 . . . $T = 3.4n - 4$

4. The third term of a quadratic sequence is 26. The seventh term is 86. The tenth term is 152.
   (a) Find an expression for the nth term of the sequence. $n^2 + 5n + 2$
   (b) Find the 16th term. 338

5. The second term of a quadratic sequence is 1. The fifth term is −17. The sixth term is −27.
   (a) Find an expression for the nth term of the sequence. $-n^2 + n + 3$
   (b) Find the 15th term. −207

6. The first term of a quadratic sequence is 8. The third term is 28. The 11th term is 348.
   (a) Find an expression for the nth term of the sequence. $3n^2 - 2n + 7$
   (b) Find the 100th term. 29803