Inductive Sequences

Remember that inductively defined sequences are not as scary as they look. Just remember to read $u_{n+1} = 2u_n + 3$ as "The next number in the sequence is twice the previous term plus three". You will also need a starting value like $u_1 = 4$. Therefore this example would yield $4, 11, 25, 53, 109, \ldots$

1. Find the first five terms in the following inductively defined sequences.

   (a) $u_{n+1} = 3u_n - 1$ with $u_1 = 7$. [$7, 20, 59, 176, 527$]

   (b) $u_{n+1} = 2 - u_n$ with $u_1 = 2$. [$2, 0, 2, 0, 2$]

   (c) $u_{n+1} = \frac{1}{u_n}$ with $u_1 = 4$. [$4, \frac{1}{4}, 4, \frac{1}{4}, 4$]

   (d) $u_{n+1} = u_n + 1$ with $u_1 = k$. [$k, k+1, k+2, k+3, k+4$]

   (e) $u_{n+1} = 2 + \frac{3}{u_n}$ with $u_1 = 1$. [$1, 5, \frac{13}{5}, \frac{41}{13}, \frac{121}{41}$]

   (f) $u_{n+1} = 2^{u_n} - 1$ with $u_1 = 2$. [$2, 3, 7, 127, 1.70 \times 10^{38}$]

   (g) $u_{n+1} = 3^{u_n} - 2^{u_n}$ with $u_1 = 2$. [$2, 5, 211, \ldots$]

   Now with the next term being dependent on the previous two terms...

   (h) $u_{n+1} = u_n + 2u_{n-1}$ with $u_1 = 1$ and $u_2 = 2$. [$1, 2, 4, 8, 16$]

   (i) $u_{n+1} = \frac{1}{u_n} + \frac{1}{u_{n-1}}$ with $u_1 = -1$ and $u_2 = 3$. [$-1, 3, -\frac{2}{3}, -\frac{7}{6}, -\frac{33}{14}$]

   (j) $u_{n+1} = \frac{1}{u_n + u_{n-1}}$ with $u_1 = -1$ and $u_2 = 3$. [$-1, 3, \frac{1}{2}, \frac{2}{7}, \frac{14}{11}$]

   (k) $u_{n+1} = u_n + u_{n-1}$ with $u_1 = a$ and $u_2 = a^2$. [$a, a^2, a^2 + a, 2a^2 + a, 3a^2 + 2a$]

2. Find the specified term in each sequence (fully simplified, of course).

   (a) $u_{n+1} = \frac{2}{u_n} + 1$ with $u_1 = 3$, find u_4. [$\frac{21}{11}$]

   (b) $u_{n+1} = \frac{2}{u_n} + 1$ with $u_1 = x$, find u_4. [$\frac{5x+6}{3x+2}$]

   (c) $u_{n+1} = \frac{1}{u_n + 1}$ with $u_1 = 3$, find u_4. [$\frac{5}{9}$]

   (d) $u_{n+1} = \frac{2}{u_n + 3}$ with $u_1 = x$, find u_4. [$\frac{6x+22}{11x+39}$]

   (e) $u_{n+1} = \frac{1+u_n}{1-u_n}$ with $u_1 = 2$, find u_5. [2]

   (f) $u_{n+1} = \frac{1+u_n}{1-u_n}$ with $u_1 = k$, find u_5. [k]

   (g) $u_{n+1} = \frac{2+u_n}{3+2u_n}$ with $u_1 = x$, find u_4. [$\frac{21x+34}{34x+55}$]

   (h) $u_{n+2} = \frac{1}{u_{n+1} + u_n}$ with $u_1 = x$ and $u_2 = y$, find u_5. [$\frac{y^3+2xy^2+x^2y+y+x}{2y^2+3xy+x^2+1}$]

3. Find the first five terms of the following sequences. I expect them to be fully cancelled down fractions (or, obviously, whole numbers).

   (a) $u_{n+1} = 2u_n - 1$ with $u_1 = 3$. [$3, 5, 9, 17, 33$]

   (b) $t_{n+1} = 3t_n + 1$ with $t_1 = \frac{1}{3}$. [$\frac{1}{3}, 2, 7, 22, 67$]

   (c) $a_{n+1} = 1 - 5a_n$ with $a_1 = 4$. [$4, -19, 96, -479, 2396$]

   (d) $u_{n+1} = \frac{1}{u_n + 2}$ with $u_1 = 1$. [$1, \frac{1}{3}, \frac{3}{7}, \frac{7}{17}, \frac{17}{41}$]

   (e) $\theta_{n+1} = \frac{\theta_n + 1}{\theta_n}$ with $\theta_1 = 1$. [$1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}$]

   (f) $\psi_{n+1} = 1 + \frac{2}{\psi_n}$ with $\psi_1 = 3$. [$3, \frac{5}{3}, \frac{11}{5}, \frac{21}{11}, \frac{43}{21}$]

   (g) i. $\alpha_{n+1} = \frac{\alpha_n - 1}{\alpha_n + 1}$ with $\alpha_1 = 5$. [$5, \frac{2}{3}, -\frac{1}{5}, -\frac{3}{2}, 5$]

   ii. What would the 123rd term of this sequence be? [$-\frac{1}{5}$]

   (h) $u_{n+1} = 1 + \frac{1}{u_n}$ with $u_1 = k$.