Single Pure - Integration Definite

1. Evaluate the following integrals, showing full working and simplifying your answer as much as possible:

   (a) $\int_{-1}^{3} x^2 + 3x - 1\, dx.$ $\boxed{\frac{52}{3}}$

   (b) $\int_{1}^{2} \frac{1}{x^2}\, dx.$ $\boxed{\frac{1}{2}}$

   (c) $\int_{0}^{k} \sqrt{x}\, dx.$ $\boxed{\frac{2k^{3/2}}{3}}$

   (d) $\int_{0}^{9} \frac{1}{\sqrt{x}}\, dx.$ $\boxed{6}$

   (e) $\int_{8}^{27} \sqrt[3]{x}\, dx.$ $\boxed{\frac{195}{4}}$

   (f) $\int_{1}^{2} x^4 - \frac{3}{x^2} + 1\, dx.$ $\boxed{\frac{57}{10}}$

   (g) $\int_{1}^{3} \frac{x^3 + x - 2}{x^5}\, dx.$ $\boxed{\frac{40}{81}}$

   (h) $\int_{0}^{\pi} \sin x\, dx.$ $\boxed{2}$

   (i) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos 2x\, dx.$ $\boxed{\frac{2-\sqrt{3}}{4}}$

   (j) $\int_{-1}^{0} e^{2x}\, dx.$ $\boxed{\frac{1}{2} - \frac{1}{2e^2}}$

   (k) $\int_{-1}^{2} (2x-3)^5\, dx.$ (Don't multiply out!) $\boxed{-1302}$

   (l) $\int_{-2}^{0} \frac{x}{\sqrt{x^2+1}}\, dx.$ $\boxed{1-\sqrt{5}}$

2. Find the value of k that satisfies the following integral equations:

   (a) $\int_{0}^{3} kx^2\, dx = 45.$ $\boxed{k=5}$

   (b) $\int_{-2}^{-1} kx^4\, dx = 62.$ $\boxed{k=10}$

   (c) $\int_{0}^{\frac{\pi}{2}} k\cos x\, dx = 5.$ $\boxed{k=5}$

   (d) $\int_{-2}^{1} kx^3\, dx = 1.$ $\boxed{k=-\frac{4}{15}}$

   (e) $\int_{0}^{1} ke^{3x}\, dx = 1.$ $\boxed{k=\frac{3}{e^3-1}}$

3. Find the value(s) of a that satisfies the following integral equations:

   (a) $\int_{0}^{a} x + 3\, dx = 20.$ $\boxed{a=4 \text{ or } a=-10}$

(b) $\int_0^a x^3 \, dx = 4.$ $a = \pm 2$

(c) $\int_0^a e^x \, dx = 1.$ $a = \ln 2$

(d) $\int_0^a \sqrt{x} \, dx = \frac{3}{2}.$ $a = \frac{27}{8}$

4. Sketch the graph of $y = 4x^2 - 9$, and find the area bounded by this curve and the x-axis. 18

5. (a) Sketch the curve $y = x^2 - 3x$.

   (b) Shade the two regions bounded by the curve and the x-axis, and the lines $x = 0$ and $x = 5$.

   (c) Explain why $\int_0^5 x^2 - 3x \, dx$ does not give the total area of the shaded regions.

   Some area below axis and some above.

   (d) Use integration to find the exact total area of the shaded regions. $\frac{35}{3}$

6. (a) Sketch (on the same set of axes) the graphs of $y = x^2 + 3x - 10$ and $y = -3x^2 - 9x + 30$. Find the area bounded by this curve and the x-axis. Show all points of intersections with the coordinate axes. $(0, -10), (0, 30), (-5, 0), (2, 0)$

   (b) By evaluating two separate integrals, find the exact area contained between the curves.

   $228\frac{2}{3}$

7. (a) Sketch the curve $y = \sin x$, where x is measured in radians, for $0 \leqslant x \leqslant 2\pi$.

   (b) Explain why $\int_0^{2\pi}$ does not find the two areas between the curve and the x-axis.

   Some area below axis and some above.

   (c) Find the sum of the two areas. 4

8. Find the areas bounded by $y = 4(x^2 - 4)(3 - x)$ and the x-axis (i) for which $y > 0$; (ii) for which $y < 0$. Write down the value of

$$\int_{-2}^{3} 4(x^2 - 4)(3 - x) \, dx.$$

9. Evaluate $\int_1^3 (4 + 3x - x^3) \, dx$. What can you deduce from your results about the graph of $y = 4 + 3x - x^3$ between $x = 1$ and $x = 3$?

10. Evaluate $\int_0^2 (x^2 + 1) \, dx$. Explain, with the aid of a sketch, how you could deduce the value of

$$\int_{-2}^{2} (x^2 + 1) \, dx.$$

Show, preferably by means of another sketch, that

$$\int_{-2}^{2} x(x^2 + 1) \, dx.$$

11. Sketch the curve $y = x(x - 2)$. Find an expression for

$$F(t) \equiv \int_{-1}^{t} x(x - 2) \, dx$$

and sketch the curve $y = F(t)$ for $-1 \leqslant x \leqslant 3$.

Find the maximum and minimum values of $F(t)$, and the values of t for which these occur. How do these relate to the first graph?

12. (a) Use the result

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F'(x) \equiv f(x)$, to show that

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

and interpret this in terms of areas when $a < b < c$.

(b) Show that putting $a = b$ in (a) suggests that

$$\int_a^a f(x)\,dx = 0,$$

and explain how this fits with the interpretation of a definite integral as an area.

(c) Show that putting $c = a$ in (a) suggests that

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

[The results in (b) and (c) are used as the definitions of $\int_a^b f(x)\,dx$ when $a \geqslant b$.]

13. Calculate the coordinates of the points where the curve $y = x^2$ crosses the line $y = 3x + 10$ and show them in a sketch of a line and the curve. On your diagram shade the area which satisfies both $y \geqslant x^2$ and $y \leqslant 3x + 10$. Calculate this area.

14. Sketch the curves $y = x^2$ and $y = 8 - x^2$, and show that they intersect at the points $(2, 4)$ and $(-2, 4)$. Calculate the area enclosed by the two curves.

15. Find the area enclosed by the curves $y = x^2$ and $y = x^3$.

16. Sketch the curves

$$y = \frac{4}{x^2},\ y = x(x+3) \text{ and } y = x - \frac{x^2}{4}.$$

for $0 \leqslant x \leqslant 2$. You *should* find they cross at $(0, 0)$, $(1, 4)$ and $(2, 1)$ and that the three curves enclose an area between these points. Find the enclosed area.

17. Show that if $n \geqslant 1$, the area enclosed by the curves $y = x^n$ and $y = x^{1/n}$ is $\dfrac{n-1}{n+1}$.

18. Sketch the curve $y = (x + 2)^2$, and shade the region defined by the inequalities $x \geqslant 0$, $y \geqslant (x + 2)^2$, $y \leqslant 9$. Calculate the area of this region.

19. Draw a rough sketch of the curve $y^2 = 8x + 8$. Calculate the area enclosed by the curve and the line $x = 1$.