OCR Core 2 Module Revision Sheet

The C2 exam is 1 hour 30 minutes long. You are allowed a graphics calculator.

Before you go into the exam make sure you are fully aware of the contents of the formula booklet you receive. Also be sure not to panic; it is not uncommon to get stuck on a question (I've been there!). Just continue with what you can do and return at the end to the question(s) you have found hard. If you have time check all your work, especially the first question you attempted... always an area prone to error.

J.M.S.

Trigonometry

- We define
$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}.$$
This identity is very useful in solving equations like $\sin\theta - 2\cos\theta = 0$ which yields $\tan\theta = 2$. The solutions of this in the range $0° \leqslant \theta \leqslant 360°$ are $\underline{\theta = 63.4°}$ and $\underline{\theta = 243.4°}$ to one decimal place.

- Know the following (or better yet, learn a couple and be able to derive the rest, quickly, from your knowledge of the trigonometric functions):

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	1	0
$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	1	0	undefined
$180°$	0	-1	0

- Be able to sketch $\sin\theta$, $\cos\theta$ and $\tan\theta$ in both degrees and radians.

- By considering a right angled triangle (or a point on the unit circle) we can derive the important result $\sin^2\theta + \cos^2\theta \equiv 1$. This is useful in solving certain trigonometric equations. Worked example; solve $1 = 2\cos^2\theta + \sin\theta$ for $0° \leqslant \theta \leqslant 360°$.
$$\begin{aligned}
1 &= 2\cos^2\theta + \sin\theta \\
1 &= 2(1-\sin^2\theta) + \sin\theta && \text{get rid of } \cos^2\theta, \\
0 &= 1 - 2\sin^2\theta + \sin\theta && \text{quadratic in } \sin\theta, \\
0 &= 2\sin^2\theta - \sin\theta - 1 && \text{factorise as normal,} \\
0 &= (2\sin\theta + 1)(\sin\theta - 1).
\end{aligned}$$
So we just solve $\sin\theta = -\frac{1}{2}$ and $\sin\theta = 1$. Therefore $\underline{\theta = 210°}$ or $\underline{\theta = 330°}$ or $\underline{\theta = 90°}$.

- The above relation is also useful in converting between the different trigonometric functions. For example if $\cos\theta = \frac{6}{7}$ then, to find $\sin\theta$, do **not** use "$\cos^{-1}$" on your calculator and then "sin" the answer. Instead

$$\sin^2\theta + \cos^2\theta = 1,$$
$$\sin^2\theta + \frac{36}{49} = 1,$$
$$\sin\theta = \pm\sqrt{\frac{13}{49}} = \pm\frac{\sqrt{13}}{7}.$$

Without further information you must keep both the positive and negative solution.

- If a question tells you that the angle is 'acute', 'obtuse' or 'reflex' then you must visualise the appropriate graph and interpret. For example given that $\sin\theta = \frac{1}{3}$ and that θ is obtuse, find the value of $\cos\theta$. By the argument above you will find that

  $$\cos\theta = \pm\frac{\sqrt{8}}{3} = \pm\frac{2\sqrt{2}}{3}.$$

  However, given an obtuse angle ($90^\circ < \theta < 180^\circ$) the cosine graph is negative, so the final answer should be $\cos\theta = -\frac{2\sqrt{2}}{3}$.

- You must be careful when you see things like $2\tan x \sin x = \tan x$. It is SO tempting to divide both sides by $\tan x$ to yield $2\sin x = 1$. But you must bring everything to one side and factorise;

  $$2\tan x\sin x - \tan x = 0 \qquad \Rightarrow \qquad \tan x(2\sin x - 1) = 0.$$

  The full set of solutions can then be by solving $\tan x = 0$ and $2\sin x - 1 = 0$. [It is completely analogous to $x^2 = x$. If we divide by x we find $x = 1$, but we know this has missed the solution $x = 0$. However when we factorise we find $x(x-1) = 0$ and both solutions are found.]

- Given a trigonometric equation it is always best first to isolate the trigonometric function on its own; for example

  $$9\cos(\dots) + 2 = 7 \qquad \Rightarrow \qquad \cos(\dots) = \tfrac{5}{9}.$$

- For complicated trigonometric equations where you are not just 'cos'ing, 'sin'ing or 'tan'ing a single variable (x, θ, t or the like), it is often easiest to make a substitution.

  For example to solve $\cos(2x+30) = \frac{1}{4}$ in the range $0^\circ \leqslant x \leqslant 360^\circ$ the desired substitution is clearly $u = 2x + 30$, but you **must** remember to also convert the range also (many students forget this) so:

  $$\cos(2x+30) = \tfrac{1}{4} \qquad 0^\circ \leqslant x \leqslant 360^\circ,$$
  $$\cos u = \tfrac{1}{4} \qquad 30^\circ \leqslant u \leqslant 750^\circ,$$
  $$u = \dots^\circ, \dots^\circ, \dots^\circ, \dots^\circ.$$

  However, we don't want solutions in u, so we need to use $x = \frac{u-30}{2}$ on each u solution to get

  $$x = \dots^\circ, \dots^\circ, \dots^\circ, \dots^\circ.$$

Sequences

- You must be comfortable with $\sum$-notation. It works as follows; you put in the number at the bottom of the $\sum$ and then keep summing until you reach the top number[1]. For example:

$$\sum_{i=4}^{8}(2i+3) = 11+13+15+17+19 = 75,$$
$$\sum_{i=1}^{n}(i^2+i) = (1+1)+(4+2)+(9+3)+\cdots+(n^2+n).$$

- A 'sequence' is a list of numbers in a specific order. A 'series' is a sum of the terms of a sequence.

- Sequences are sometimes defined *recursively*. For example the sequence $a_{n+1} = a_n + 3$ with $a_1 = 10$ defines the sequence $10, 13, 16, 19\ldots$ We know that this is an arithmetic sequence which can also be defined *deductively* by $a_n = 10 + 3(n-1)$.

- An arithmetic sequence increases or decreases by a constant amount. The letter a always denotes the first term and d is the difference between the terms (negative for a decreasing sequence!). The nth term is denoted a_n and satisfies the important relationship

$$a_n = a + (n-1)d.$$

For example if told the third term of a sequence is 10 and the seventh term is 34 then we can use the above equation to find the a and d.

$$\begin{array}{l} 10 = a + (3-1)d \\ 34 = a + (7-1)d \end{array} \quad \Rightarrow \quad 4d = 24 \quad \Rightarrow \quad d = 6 \quad \Rightarrow \quad a = -2.$$

- The sum of the n terms of an arithmetic sequence is given by

$$S = \frac{n}{2}(\text{First} + \text{Last}) = \frac{n}{2}(2a + (n-1)d).$$

For example the sum of the first 10 terms of a sequence is 130 and the first term is 4. What is the difference?

$$S = \frac{n}{2}(2a + (n-1)d) \quad \Rightarrow \quad 130 = \frac{10}{2}(8 + (10-1)d) \quad \Rightarrow \quad d = 2.$$

Binomial Theorem

- Binomial expansion allows us to expand $(a+b)^n$ for any integer n. Best explained by means of an example; expand $(2x-y)^5$.

1. Begin by considering 'prototype' expansion of $(a+b)^5$.
2. So $(a+b)^5 = \binom{5}{0}a^5 + \binom{5}{1}a^4b + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^3 + \binom{5}{4}ab^4 + \binom{5}{5}b^5$.
3. Calculate binomial coefficients either on calculator or by drawing a mini Pascal's Triangle to give $(a+b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$.

[1] If you want to multiply instead then use $\prod$-notation. For example $\prod_{i=1}^{n} i = 1 \times 2 \times 3 \times 4 \times \cdots \times n \equiv n!$

4. Next notice that in our case $a = 2x$ and $b = -y$ and substitute in to get $(2x - y)^5 = 1(2x)^5 + 5(2x)^4(-y) + 10(2x)^3(-y)^2 + 10(2x)^2(-y)^3 + 5(2x)(-y)^4 + 1(-y)^5$.
5. Tidying up we get $(2x - y)^5 = 32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5$.

- It is worth noting that when the expansion is of the form (something – another thing)^n, then the signs will alternate.

- Also of note is the way each *individual* component is constructed. For example; find the x^5 coefficient in the expansion of $(2 - 3x)^7$. The component with x^5 is given by $\binom{7}{5}(2)^2(-3x)^5 = -20412x^5$, so the coefficient is -20412.

- $\binom{n}{r} = {}^nC_r = \dfrac{n!}{r!(n-r)!}$. For example $\binom{5}{2} = \dfrac{5!}{2!\,3!} = \dfrac{5 \times 4 \times 3 \times 2 \times 1}{(2 \times 1) \times (3 \times 2 \times 1)} = 10$.

- Know that $(1 + x)^n$ expands thus:

$$\begin{aligned}(1+x)^n &= 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + x^n \\ &= 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n.\end{aligned}$$

(This is particularly useful in Core 4.)

Sine & Cosine Rules

- The sine rule states for any triangle $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

- The cosine rule states that $a^2 = b^2 + c^2 - 2bc\cos A$. Practice both sine and cosine rules on page 293.

- By considering half of a general parallelogram we can show that the area of any triangle is given by $A = \frac{1}{2}ab\sin C$.

- You must be good at bearing problems which result in triangles. Remember to draw lots of North lines and remember also that they are all parallel; therefore you can use Corresponding, Alternate and Allied angle theorems. . . revise your GCSE notes! Bearings are measured clockwise from North and must contain three digits. For example

$$12.2^\circ \quad \Rightarrow \quad 012.2^\circ.$$

Integration

- *Calculus* is the combined study of differentiation *and* integration (and their relationship). A good description is that calculus is the study of change in the same way that geometry is the study of shapes.

- Integration is the reverse of differentiation. That is if $\frac{dy}{dx} = f(x)$ then $y = \int f(x)\,dx$. For example if $\frac{dy}{dx} = 3x^3$ then $y = \int 3x^3\,dx = \frac{3}{4}x^4 + c$.

- The general rule is therefore $\displaystyle\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$.

- $\int y\,dx$ is an *indefinite* integral because there are no limits on the integral sign. When evaluating these integrals *never* forget an *arbitrary constant* added on at the end. For example $\int 6x^2\,dx = 2x^3 + c$.

- $\int_a^b y\,dx$ is a *definite integral* and is the area between the curve and the x-axis from $x = a$ to $x = b$. Areas under the x-axis are negative. (For areas between the curve and the y-axis switch the x and the y and use $\int_p^q x\,dy$ between $y = p$ and $y = q$.)
- To find the area *between* two curves between $x = a$ and $x = b$ evaluate
$$\int_a^b (\text{top} - \text{bottom})\,dx.$$
- For example, given that $y = \sqrt{x} + \frac{1}{\sqrt{x}}$, find the area under the curve from $x = 1$ and $x = 2$.
$$\begin{aligned}\int_1^2 \sqrt{x} + \frac{1}{\sqrt{x}}\,dx &= \int_1^2 x^{\frac{1}{2}} + x^{-\frac{1}{2}}\,dx\\ &= \left[\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}}\right]_1^2\\ &= \left(\frac{2}{3} \times 2^{\frac{3}{2}} + 2 \times 2^{\frac{1}{2}}\right) - \left(\frac{2}{3} + 2\right)\\ &= \frac{10\sqrt{2}}{3} - \frac{8}{3}.\end{aligned}$$
- To calculate integrals where one of the limits is infinite (∞ or $-\infty$), proceed as normal until you input the ∞ or $-\infty$ into the integral. Then you must **not** write such things as
$$\frac{1}{\infty}, \quad \frac{1}{3^\infty}, \quad \frac{2\infty + 1}{3\infty - 2}, \quad 2^{\frac{1}{\infty}}, \qquad \text{and the like.}$$
You must think what these things would equal and just write down the number; in the previous four cases you would get 0, 0, $\frac{2}{3}$ and 1. (If, in the C2 exam, you think that when you put in ∞ you get an infinite answer, chances are you've made a mistake somewhere.) For example:
$$\begin{aligned}\int_2^\infty \frac{8}{5x^3}dx &= \int_2^\infty \frac{8}{5}x^{-3}dx\\ &= \left[-\frac{4}{5}x^{-2}\right]_2^\infty\\ &= (0) - \left(-\frac{4}{5} \times \frac{1}{2^2}\right) = \frac{1}{5}.\end{aligned}$$

Geometric Sequences

- A geometric sequence is one where the terms are multiplied by a constant amount. For example $1, 2, 4, 8, 16, \ldots, [2^{n-1}]$ is a geometric sequence with $a = 1$ and $r = 2$. The n^{th} term is given by
$$a_n = ar^{n-1}.$$
So for the above example the 20th term is $a_{20} = 1 \times 2^{19} = 524288$.
- The sum of n terms of a geometric sequence is given by
$$S = a\left(\frac{r^n - 1}{r - 1}\right) \qquad \text{or (equivalently) by} \qquad S = a\left(\frac{1 - r^n}{1 - r}\right).$$
For example sum the first 20 terms of $4, 2, 1, \frac{1}{2}, \ldots, [4 \times \left(\frac{1}{2}\right)^{n-1}]$. This is given by
$$S = 4\left(\frac{\left(\frac{1}{2}\right)^{20} - 1}{\frac{1}{2} - 1}\right) = 7.999992371\ldots$$

- If the ratio (r) lies between -1 and 1 (i.e. $-1 < r < 1$) then there exists a 'sum to infinity' given by
$$S_\infty = \frac{a}{1-r}.$$
Therefore S_∞ for the above example is $S_\infty = \frac{4}{1-\frac{1}{2}} = 8$. We can see that the sum to 20 terms is very close to S_∞.

Exponentials & Logarithms

- (In these notes if I write $\log x$ I *mean* $\log_{10} x$. If I mean a different base, I will write it *explicitly* as $\log_a x$. When we see $\log_a x$ we say "log to the base a of x".)
- We *define* a logarithm to be the solution (in x) to the equation $a^x = b$. It is written $x = \log_a b$. The fundamental relationship is therefore
$$a^x = b \qquad \Leftrightarrow \qquad x = \log_a b. \qquad\qquad \dagger$$
- From † we see that $\log_a b$ *means*: "The number a has to be raised to, to make b". Therefore some simple logarithms can be calculated without a calculator:
$$\begin{aligned} \log_2 8 &= \text{“the number 2 has to be raised to, to make 8”} &&= 3,\\ \log_{10} 10000 &= \text{“the number 10 has to be raised to, to make 10000”} &&= 4,\\ \log_9 3 &= \text{“the number 9 has to be raised to, to make 3”} &&= \tfrac{1}{2}, \quad (\because 3 = \sqrt{9} = 9^{\frac{1}{2}})\\ \log_a a &= \text{“the number } a \text{ has to be raised to, to make } a\text{”} &&= 1. \end{aligned}$$
- We see from † that logarithms 'pluck out powers' from equations. Therefore if you ever see an equation with the unknown in the power, then that is the clue that you will need to use logarithms. For example to solve $7^{2x-1} = 22$ we discover
$$\begin{aligned} 7^{2x-1} &= 22,\\ 2x-1 &= \log_7 22,\\ x &= \tfrac{1}{2}\log_7 22 + \tfrac{1}{2}. \end{aligned}$$
However we need to build on † because not all equations are this simple (e.g. $3 \times 2^{2x-1} = 5 \times 7^{x+1}$) and not all calculators can calculate $\log_7 22$.
- You can also use † to eliminate logarithms from an equation. Given an equation of the form $\log_a(\cdots) = b$, you can eliminate the logarithm instantly to get $(\cdots) = a^b$. A good way to remember this[2] is 'Girvan's Bullying Base'. So if we have $\log_3 x = 8$, then the bullying base '3' knocks the log out of the way and moves to the other side and squeezes up the 8 to put it in its place; therefore $x = 3^8$.
- Logarithms and exponentials (powers) are the inverse functions of each other (as can be seen from † if one puts one into the other). Therefore
$$\log_{10} 10^x = x \quad \text{and} \quad 10^{\log_{10} x} = x.$$
So if $\log a = 5.4$ then $a = 10^{5.4}$.

[2]From a colleague I respect; Mr Girvan

- There are some rules that can be derived from † that *must* be learnt. They are (for all bases):

$$\begin{aligned} \log(ab) &= \log a + \log b & \log 1 &= 0 \\ \log(\frac{a}{b}) &= \log a - \log b & \log_a a &= 1 \\ \log(a^n) &= n \log a & \log_a b &= \frac{\log_c b}{\log_c a}. \\ \log(\frac{1}{a}) &= -\log a \end{aligned}$$

- When we need to solve an equation where the unknown is in the exponent such as $5^{2x-1} = 8$ take $\log_{10}$ of both sides and simplify:

$$\begin{aligned} 5^{2x-1} &= 8 \\ \log_{10}(5^{2x-1}) &= \log_{10} 8 \\ (2x-1)\log_{10} 5 &= \log_{10} 8 \\ 2x - 1 &= \frac{\log_{10} 8}{\log_{10} 5} \\ x &= \frac{1}{2} \times \left(\frac{\log_{10} 8}{\log_{10} 5} + 1\right) \\ x &= 1.15 \text{ (3sf)}. \end{aligned}$$

Factors and Remainders

- Need to know how to divide any polynomial by a linear factor of the form $ax - b$. For example divide $x^3 + 2x^2 + 3x - 6$ by $x - 2$. (*Always* devote a column to each power of x.)

$$\begin{array}{r|rrrr} & & x^2 & +4x & +11 \\ \hline x-2 & +x^3 & +2x^2 & +3x & -6 \\ & +x^3 & -2x^2 & & \\ \hline & & +4x^2 & +3x & \\ & & +4x^2 & -8x & \\ \hline & & & +11x & -6 \\ & & & +11x & -22 \\ \hline & & & & +16 \end{array}$$

So the remainder is +16. Therefore $x^3 + 2x^2 + 3x - 6 = (x-2)(x^2 + 4x + 11) + 16$.

- If the remainder is zero, then the divisor is said to be a *factor* of the original polynomial.
- The Factor Theorem states:

$$(x - a) \text{ is a factor of } f(x) \quad \Leftrightarrow \quad f(a) = 0.$$

[More generally (but used less often in exams) is:

$$(ax - b) \text{ is a factor of } f(x) \quad \Leftrightarrow \quad f(\tfrac{b}{a}) = 0.]$$

- For example if $x^3 + ax^2 + 8x - 4$ has $(x - 2)$ as a factor, find a. From factor theorem we know $f(2) = 0$, so we discover $2^3 + a \times 2^2 + 8 \times 2 - 4 = 0$, and therefore $a = -5$.

- The Remainder Theorem states:

  When $f(x)$ is divided by $(x-a)$ the remainder is $f(a)$.

  [More generally (but used less often in exams) is:

  When $f(x)$ is divided by $(ax-b)$ the remainder is $f(\frac{b}{a})$.]

  Notice that the factor theorem is a subset of the remainder theorem. In the factor theorem all remainders are zero, by definition.

- For example if told that when $f(x) = x^3 + 2x^2 - 3x - 7$ is divided by $x - 2$ the remainder is 3, we know $f(2) = 3$.

- Worked example: $f(x) = 2x^3 + 3x^2 + kx - 2$. The remainder when $f(x)$ is divided by $(x-2)$ is four times the remainder when $f(x)$ is divided by $(x+1)$. Find k. We know

$$\begin{aligned} f(2) &= 4 \times f(-1) \\ 2 \times 2^3 + 3 \times 2^2 + 2k - 2 &= 4[2 \times (-1)^3 + 3 \times (-1)^2 - k - 2] \\ k &= -5. \end{aligned}$$

Radians

- There are (by definition) 2π radians in a circle. So $360° = 2\pi$. To convert from degrees to radians we use the conversion factor of $\frac{\pi}{180}$. For example to convert $45°$ to radians we calculate $45 \times \frac{\pi}{180} = \frac{\pi}{4}$ rad. From radians to degrees we use its reciprocal $\frac{180}{\pi}$.

- *When using radians* the formulae for arc length and area of a sector of a circle become simpler. They are $s = r\theta$ and $A = \frac{1}{2}r^2\theta$.

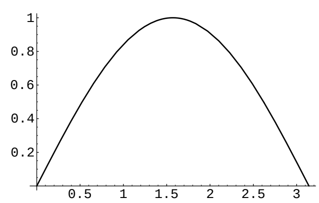
The Trapezium Rule

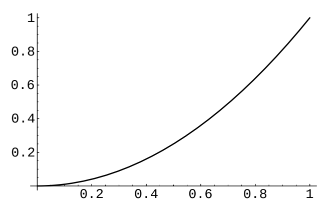
- The area under *any* curve can be *approximated* by the Trapezium Rule. The governing formula is given by (and contained in the formula booklet you will have in the exam)

$$\int_a^b y\,dx \approx \frac{1}{2}h\left[y_0 + y_n + 2(y_1 + y_2 + \cdots + y_{n-1})\right],$$

  where h is the width of each trapezium, y_0 and y_n are the 'end' heights and $y_1 + y_2 + \cdots + y_{n-1}$ are the 'internal' heights.

- By considering the shape of the graph in the interval over which you are approximating it should be clear whether your estimate of the area is an over or under-estimate of the *true* area.





  For example if you were to estimate $\int_0^\pi \sin x\,dx$ (above, left) using the trapezium rule, due to the shape of the curve, the trapezia would all fall below the curve, so we would obtain an *under*-estimate. However, with $\int_0^1 x^2\,dx$ (above, right) we would obtain an *over*-estimate.