

2n Pairs Problem

Problem

How many different ways are there of pairing $2n$ people?

Solution

Firstly we note that if there are 4 players then there are clearly 3×1 possible pairs. Also with 6 players there are $5 \times 3 \times 1$. This argument easily extends to $2n$ players resulting in $(2n - 1) \times (2n - 3) \times \cdots \times 5 \times 3 \times 1$ possible pairs. So we need to simplify this.

Next we notice that

$$[(2n - 1) \times (2n - 3) \times \cdots \times 5 \times 3 \times 1] \times [(2n - 2) \times \cdots \times 4 \times 2] = (2n - 1)!$$

Therefore

$$\begin{aligned} \text{Number of pairs} &= (2n - 1) \times (2n - 3) \times \cdots \times 5 \times 3 \times 1 \\ &= \frac{(2n - 1)!}{(2n - 2) \times \cdots \times 4 \times 2} \\ &= \frac{(2n - 1)!}{2^{n-1}((n - 1) \times \cdots \times 2 \times 1)} \\ &= \frac{(2n - 1)!}{2^{n-1}(n - 1)!} \end{aligned}$$

As required.

A much better result (in my opinion, and what I did first) is this:

We notice that

$$[(2n - 1) \times (2n - 3) \times \cdots \times 5 \times 3 \times 1] \times [(2n) \times \cdots \times 4 \times 2] = (2n)!$$

Therefore

$$\begin{aligned} \text{Number of pairs} &= (2n - 1) \times (2n - 3) \times \cdots \times 5 \times 3 \times 1 \\ &= \frac{(2n)!}{(2n) \times \cdots \times 4 \times 2} \\ &= \frac{(2n)!}{2^n(n \times \cdots \times 2 \times 1)} \\ &= \frac{(2n)!}{2^n n!} \end{aligned}$$