Logarithms Theory

Beginnings

One way of thinking about logarithms is that they allow us to solve a type of equation we couldn't before. For the following types of equations we know how to solve the first four by the operations of $+$, $-$, $\times$ and $\div$.

$x - a = b$	Addition	$x = b + a$
$x + a = b$	Subtraction	$x = b - a$
$\frac{x}{a} = b$	Multiplication	$x = ab$
$ax = b$	Division	$x = \frac{b}{a}$
$a^x = b$	?	?

Clearly we can solve the last equation in simple cases ($2^x = 16$ implies $x = 4$) but for harder cases we are at a loss (for example $3^x = 8$). So we *define* the solution to be

$$3^x = 8 \qquad \Leftrightarrow \qquad x = \log_3 8.$$

We read this as 'log to the base 3 of 8'. The value of $\log_3 8$ represents **the number we have to raise 3 to to obtain 8**. In general we have

$$a^x = b \qquad \Leftrightarrow \qquad x = \log_a b.$$

If we fiddle with the above line a little we find that

$$a^{\log_a b} = b \qquad \text{and} \qquad x = \log_a(a^x).$$

Therefore we see the exponential a^x is the inverse operation of $\log_a x$ in the same way that $+$ is the inverse operation of $-$.

The Rules

There are a few rules that you *must* learn about logarithms. They are:

$$\log_a(bc) = \log_a b + \log_a c \qquad \log_a 1 = 0$$

$$\log_a\left(\frac{b}{c}\right) = \log_a b - \log_a c \qquad \log_a a = 1$$

$$\log_a(b^n) = n\log_a b \qquad \log_a b = \frac{\log_c b}{\log_c a}$$

$$\log_a\left(\frac{1}{b}\right) = -\log_a b$$

When we need to solve an equation such as $5^x = 7$ we know that the solution is $x = \log_5 7$. However, our calculator has only got $\log_{10}$ so what we do is take $\log_{10}$ of both sides and simplify:

$$\begin{aligned} 5^x &= 7 \\ \log_{10}(5^x) &= \log_{10} 7 \\ x\log_{10} 5 &= \log_{10} 7 \\ x &= \frac{\log_{10} 7}{\log_{10} 5}. \end{aligned}$$