Integration

Whereas differentiation is concerned with finding the gradient of a curve at a given point, *integration* is concerned with finding the area under a curve between two points.

From First Principles

Consider the curve $y = kx^2$ for $k > 0$ and let us try and find the area under the graph between $x = 0$ and $x = a$. We understand the concept of area from our studies of rectangles, triangle and circles, but for the curve $y = kx^2$ none of the shapes mentioned work. So let us instead consider an approximation by rectangles. We can see that the more rectangles we use, the better the approximation gets.

If we have n rectangles under the graph then the width of each rectangle is a/n. So the area of all the rectangles is for base b and height h_n

$$\begin{aligned}
\text{Area of Rectangles} &= (b \times h_1 + b \times h_2 + b \times h_3 + \cdots + b \times h_n) \\
&= b(h_1 + h_2 + h_3 + \cdots + h_n) \\
&= \frac{a}{n}\left(k(0)^2 + k\left(\frac{a}{n}\right)^2 + k\left(2\frac{a}{n}\right)^2 + \cdots + k\left((n-1)\frac{a}{n}\right)^2\right) \\
&= \frac{a}{n}k\left(\frac{a}{n}\right)^2\left(0^2 + 1^2 + 2^2 + \cdots + (n-1)^2\right)
\end{aligned}$$

Now it can be shown[1] that $0^2 + 1^2 + 2^2 + \cdots + (n-1)^2$ is equal to $\frac{1}{6}(n-1)n(2n-1)$ so we can tidy our equation for Area up to

$$\begin{aligned}
\text{Area of Rectangles} &= \frac{a}{n}k\left(\frac{a}{n}\right)^2\left(\frac{1}{6}(n-1)n(2n-1)\right) \\
&= k\frac{a^3}{n^3}\left(\frac{1}{6}(2n^3 - 3n^2 + n)\right) \\
&= k\frac{a^3}{6}\left(2 - \frac{3}{n} + \frac{1}{n^2}\right)
\end{aligned}$$

Now comes the clever bit! If we let the number of rectangles tend to infinity we should have a perfect measure of the area under the graph. We can do this by letting n tend to infinity. Now the area under a function $f(x)$ between $x = a$ and $x = b$ with $b > a$ is denoted by

$$\int_a^b f(x)dx$$

So

$$\begin{aligned}
\text{Area under } kx^2 \text{ between } 0 \text{ and } a = \int_0^a kx^2 dx &= \lim_{n\to\infty} k\frac{a^3}{6}\left(2 - \frac{3}{n} + \frac{1}{n^2}\right) \\
&= k\frac{a^3}{6}\lim_{n\to\infty}\left(2 - \frac{3}{n} + \frac{1}{n^2}\right) \\
&= k\frac{a^3}{6} \times 2 = k\frac{a^3}{3}
\end{aligned}$$

Therefore

$$\boxed{\int_0^a kx^2 dx = k\frac{a^3}{3}}$$

[1] indeed, it will be shown in *FP1*.

Now if $b > a > 0$ then the area between a and b is

$$\int_a^b kx^2dx = \int_0^b kx^2dx - \int_0^a kx^2dx = k\frac{b^3}{3} - k\frac{a^3}{3}$$

Now instead of having a and b as fixed limits let's consider the integral between a fixed lower limit and a variable upper limit x. But before we do this we should note that our choice of variable in the integral is completely arbitrary. That is

$$\int_a^b kx^2dx = \int_a^b kt^2dt = \int_a^b k\theta^2d\theta$$

And we should also note that we *cannot* write

$$\int_0^x kx^2dx$$

We must use a *dummy variable* instead when considering variable limits. Therefore we write

$$\int_a^x ky^2dy = k\frac{x^3}{3} - k\frac{a^3}{3}.$$

Integration and Differentiation

Consider what happens if we differentiate ???.

$$\begin{aligned}
\frac{d}{dx}\left(k\frac{x^3}{3} - k\frac{a^3}{3}\right) &= \frac{k}{3}\frac{d}{dx}\left(x^3\right) \qquad \text{remembering that } a \text{ is constant} \\
&= \frac{k}{3} \times 3 \times x^2 \\
&= kx^2
\end{aligned}$$

Giving the result

$$\frac{d}{dx}\left(\int_a^x ky^2dy\right) = kx^2.$$

This suggests that differentiation and integration are inverse functions of each other. This guess is in fact correct and is the *Fundamental Theorem of Calculus* which states

$$\boxed{\frac{d}{dx}\left(\int_a^x f(y)dy\right) = f(x)}$$