OCR
Oxford Cambridge and RSA

Wednesday 29 June 2016 – Morning

A2 GCE MATHEMATICS

4731/01 Mechanics 4

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4731/01
- List of Formulae (MF1)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.
- The Question Paper will be found inside the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book**. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION FOR CANDIDATES

- This information is the same on the Printed Answer Book and the Question Paper.
- The number of marks is given in brackets **[ ]** at the end of each question or part question on the Question Paper.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **12** pages. The Question Paper consists of **8** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

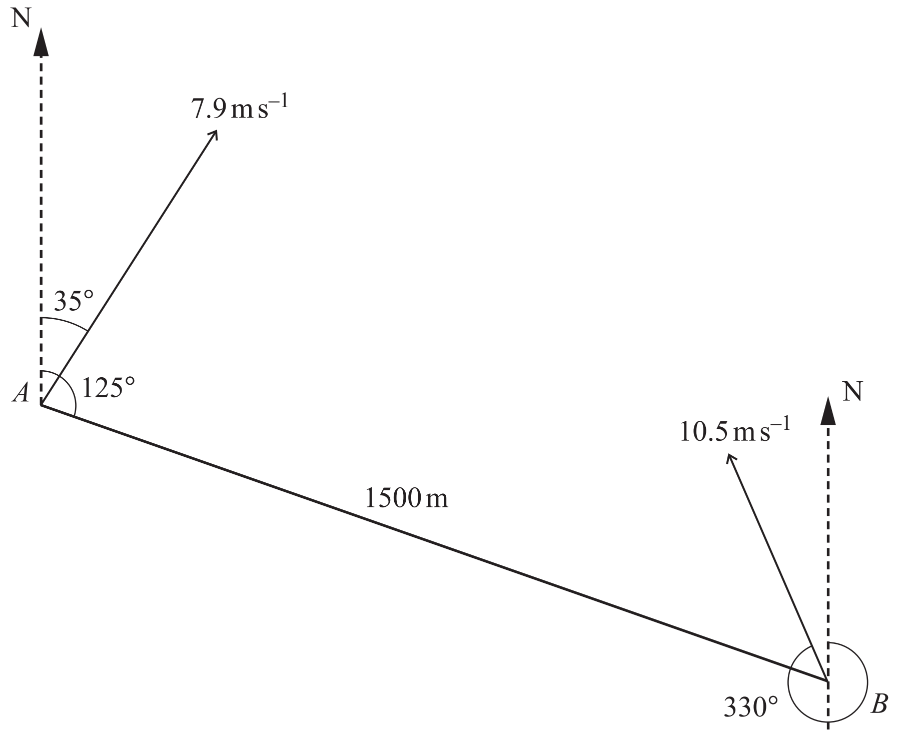
- Do not send this Question Paper for marking; it should be retained in the centre or recycled. Please contact OCR Copyright should you wish to re-use this document.

Turn over

Answer **all** the questions.

1 A uniform square lamina, of mass 5 kg and side 0.2 m, is rotating about a fixed vertical axis that is perpendicular to the lamina and that passes through its centre. A couple of constant moment 0.06 N m is applied to the lamina. The lamina turns through an angle of 155 radians while its angular speed increases from $8\,\text{rad s}^{-1}$ to $\omega\,\text{rad s}^{-1}$. Find ω. **[4]**

2



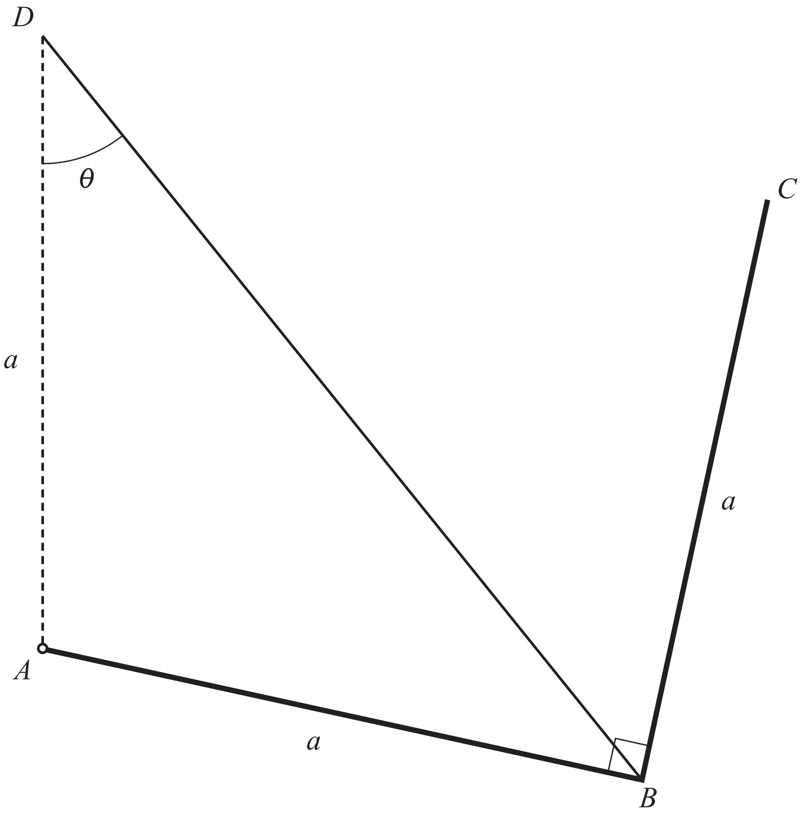
Boat A is travelling with constant speed $7.9\,\text{m s}^{-1}$ on a course with bearing 035°. Boat B is travelling with constant speed $10.5\,\text{m s}^{-1}$ on a course with bearing 330°. At one instant, the boats are 1500 m apart with B on a bearing of 125° from A (see diagram).

(i) Find the magnitude and the bearing of the velocity of B relative to A. **[5]**

(ii) Find the shortest distance between A and B in the subsequent motion. **[2]**

(iii) Find the time taken from the instant when A and B are 1500 m apart to the instant when A and B are at the point of closest approach. **[2]**

3



Two uniform rods AB and BC, each of length a and mass m, are rigidly joined together so that AB is perpendicular to BC. The rod AB is freely hinged to a fixed point at A. The rods can rotate in a vertical plane about a smooth fixed horizontal axis through A. One end of a light elastic string of natural length a and modulus of elasticity λmg is attached to B. The other end of the string is attached to a fixed point D vertically above A, where $AD = a$. The string BD makes an angle θ radians with the downward vertical (see diagram).

(i) Taking D as the reference level for gravitational potential energy, show that the total potential energy V of the system is given by

$$V = \tfrac{1}{2}mga(\sin 2\theta - 3\cos 2\theta) + \tfrac{1}{2}\lambda mga(2\cos\theta - 1)^2 - 2mga.$$

[5]

(ii) Given that $\theta = \frac{1}{4}\pi$ is a position of equilibrium, find the exact value of λ. **[4]**

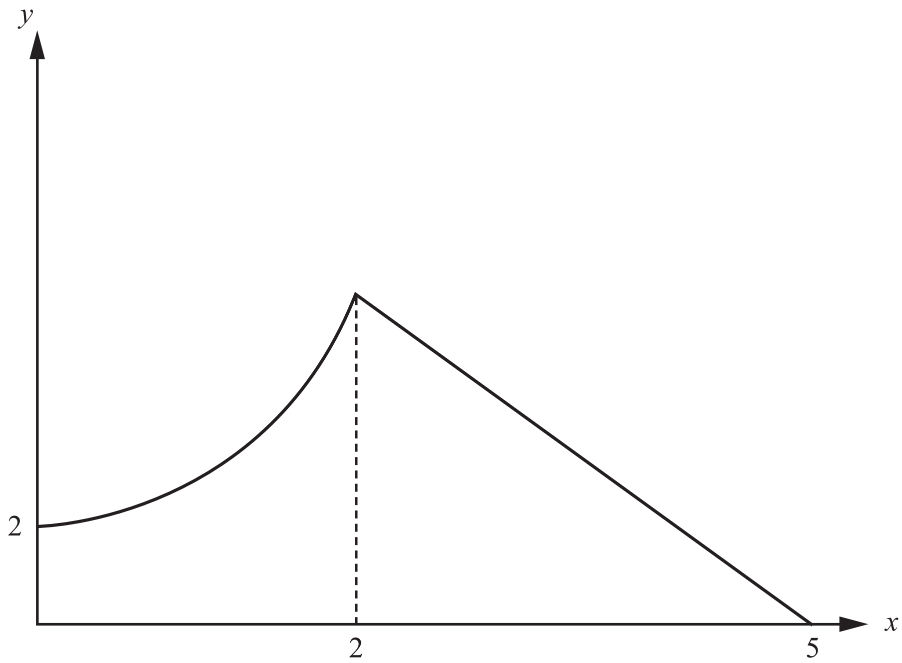
(iii) Find $\dfrac{\mathrm{d}^2 V}{\mathrm{d}\theta^2}$ and hence determine whether the position of equilibrium at $\theta = \frac{1}{4}\pi$ is stable or unstable. **[4]**

4 The region bounded by the curve $y = 2e^{\frac{1}{2}x}$ for $0 \leqslant x \leqslant 2$, the x-axis, the y-axis and the line $x = 2$, is occupied by a uniform lamina.

(i) Find the exact value of the y-coordinate of the centre of mass of the lamina. **[6]**

As shown in the diagram below, a uniform lamina occupies the closed region bounded by the x-axis, the y-axis and the curve $y = f(x)$ where

$$f(x) = \begin{cases} 2e^{\frac{1}{2}x} & 0 \leqslant x \leqslant 2, \\ \frac{2}{3}(5-x)e & 2 \leqslant x \leqslant 5. \end{cases}$$



(ii) Find the exact value of the x-coordinate of the centre of mass of the lamina. **[7]**

5 A uniform rod AB has mass $2m$ and length $4a$.

(i) Show by integration that the moment of inertia of the rod about an axis perpendicular to the rod through A is $\frac{32}{3}ma^2$ **[4]**

The rod is initially at rest with B vertically below A and it is free to rotate in a vertical plane about a smooth fixed horizontal axis through A. A particle of mass m is moving horizontally in the plane in which the rod is free to rotate. The particle has speed v, and strikes the rod at B. In the subsequent motion the particle adheres to the rod and the combined rigid body Q, consisting of the rod and the particle, starts to rotate.

(ii) Find, in terms of v and a, the initial angular speed of Q. **[4]**

At time t seconds the angle between Q and the downward vertical is θ radians.

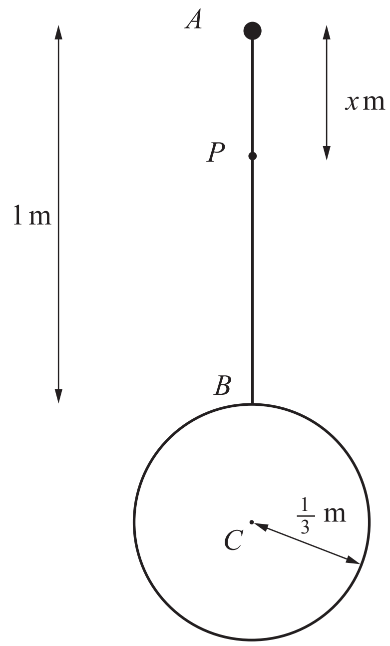
(iii) Show that $\dot{\theta}^2 = k\frac{g}{a}(\cos\theta - 1) + \frac{9v^2}{400a^2}$, stating the value of the constant k. **[4]**

(iv) Find, in terms of a and g, the set of values of v^2 for which Q makes complete revolutions. **[2]**

When Q is horizontal, the force exerted by the axis on Q has vertically upwards component R.

(v) Find R in terms of m and g. **[4]**

6



A compound pendulum consists of a uniform rod AB of length 1 m and mass 3 kg, a particle of mass 1 kg attached to the rod at A and a circular disc of radius $\frac{1}{3}$ m, mass 6 kg and centre C. The end B of the rod is rigidly attached to a point on the circumference of the disc in such a way that ABC is a straight line. The pendulum is initially at rest with B vertically below A and it is free to rotate in a vertical plane about a smooth fixed horizontal axis passing through the point P on the rod where $AP = x$ m and $x < \frac{1}{2}$ (see diagram).

(i) Show that the moment of inertia of the pendulum about the axis of rotation is $(10x^2 - 19x + 12)$ kg m^2. **[6]**

The pendulum is making small oscillations about the equilibrium position, such that at time t seconds the angular displacement that the pendulum makes with the downward vertical is θ radians.

(ii) Find the angular acceleration of the pendulum, in terms of x, g and θ. **[4]**

(iii) Show that the motion is approximately simple harmonic, and show that the approximate period of oscillations, in seconds, is given by $2\pi\sqrt{\dfrac{20x^2 - 38x + 24}{(19 - 20x)g}}$. **[2]**

(iv) Hence find the value of x for which the approximate period of oscillations is least. **[3]**

END OF QUESTION PAPER

Question			Answer	Marks	Guidance	
1			$I=\frac{1}{3}(5)(0.1^2+0.1^2)\left(=\frac{1}{30}\right)$	B1		Work-energy principle (M1 A1ft)
			$\alpha=\frac{0.06}{\left(\frac{1}{30}\right)}(=1.8)$	M1	M1 for using $C=I\alpha$ with their I	$\frac{1}{2}\left(\frac{1}{30}\right)\omega^2-\frac{1}{2}\left(\frac{1}{30}\right)8^2=155(0.06)$
			$\omega^2=8^2+2(1.8)(155)$	M1	Using $\omega^2=\omega_0^2+2\alpha\theta$ with their α	
			$\omega=24.9$ (3 sf)	A1	accept $\sqrt{622}$	24.939927…
				[4]		
2		(i)				$w_v=10.5\cos 30-7.9\cos 35=2.62...$
			$w^2=7.9^2+10.5^2-2(7.9)(10.5)\cos(30+35)$	M1	Use of cosine rule	$w_h=10.5\sin 30+7.9\sin 35=9.78...$
			$w=10.1$	A1		10.12658…
			$\frac{\sin\theta}{7.9}=\frac{\sin(30+35)}{10.12658...}$ or $\frac{\sin\alpha}{10.5}=\frac{\sin(30+35)}{10.12658...}$	M1	Use of sine rule with their w	$\tan\beta=\frac{w_h}{w_v}$ or $\tan\gamma=\frac{w_h}{w_v}$
			$\theta=45.0$ or $\alpha=70.0$	A1		44.99406… ($\beta=15.005...$ or $\gamma=74.994...$)
			Bearing = $330-\theta=285^\circ$ or $180+35+\alpha=285^\circ$	A1		285.0059…
				[5]		
		(ii)	Shortest distance = $d=1500\sin(44.99406...-25)$	M1	$1500\sin\beta$	
			$d=513$ (3 sf)	A1		512.8841…
				[2]		
		(iii)	$t=\frac{1500\cos(44.99406...-25)}{10.12658...}$	M1	Use of $s=ut$ with their w and θ	$\frac{1500\cos\beta}{w}$ or $\frac{\sqrt{1500^2-d^2}}{w}$
			= 139 (3 sf)	A1		139.1972… (β consistent with (ii))
				[2]		

Question			Answer	Marks	Guidance
3		(i)	BD = $2a\cos\theta$	B1	Award if seen in EPE term
			GPE for rod AB = $(-)mg\left(a+\frac{1}{2}a\cos 2\theta\right)$	B1	$(-)mg\left(a+\frac{1}{2}a\sin(90-2\theta)\right)$
			GPE for rod BC = $(-)mg\left(a+a\cos 2\theta-\frac{1}{2}a\sin 2\theta\right)$	B1	$(-)mg\left(a+a\sin(90-2\theta)-\frac{1}{2}a\cos(90-2\theta)\right)$
			EPE $=\dfrac{\lambda mg(2a\cos\theta-a)^2}{2a}$	M1	Using $\dfrac{\lambda x^2}{2a}$ with their BD
			$V=\frac{1}{2}mga(\sin 2\theta-3\cos 2\theta)+\frac{1}{2}\lambda mga(2\cos\theta-1)^2-2mga$	A1 **[5]**	**AG** correctly shown
		(ii)	$\dfrac{dV}{d\theta}=\frac{1}{2}mga(2\cos 2\theta+6\sin 2\theta)+$	M1	Attempt at differentiation
			$\lambda mga(2\cos\theta-1)(-2\sin\theta)$	A1	Correct derivative $\ldots+\lambda mga(2\sin\theta-2\sin 2\theta)$
			$\frac{1}{2}(2(0)+6)+\lambda\left(\frac{2\sqrt{2}}{2}-1\right)\left(-\frac{2\sqrt{2}}{2}\right)=0$	M1	Set $\dfrac{dV}{d\theta}=0$ and $\theta=\frac{1}{4}\pi$
			$\lambda=\dfrac{3}{2-\sqrt{2}}$	A1 **[4]**	oe eg $\dfrac{6+3\sqrt{2}}{2}$
		(iii)	$\dfrac{d^2V}{d\theta^2}=\frac{1}{2}mga(-4\sin 2\theta+12\cos 2\theta)$	M1	Attempt at second derivative
			$-2\lambda mga(2\cos 2\theta-\cos\theta)$	A1	oe $\ldots-2\lambda mga\left[(2\cos\theta-1)\cos\theta-2\sin^2\theta\right]$
				M1	Set $\theta=\frac{1}{4}\pi$ and using their λ (must be evidence of substitution)
			$\dfrac{d^2V}{d\theta^2}=mga(1+3\sqrt{2})>0$, so equilibrium is stable	A1 **[4]**	Correct value of V'' and > 0 $V''=(5.24264\ldots)mga$

Question			Answer	Marks	Guidance	
4		(i)	$A_1 = \int_0^2 2e^{\frac{1}{2}x} dx$	M1*	Attempt at integration to find area	Limits not required for M mark
			$= \left[4e^{\frac{1}{2}x}\right]_0^2 = 4(e-1)$	A1		
			$A_1\bar{y} = \frac{1}{2}\int_0^2 \left(2e^{\frac{1}{2}x}\right)^2 dx = \frac{1}{2}\int_0^2 4e^x dx$	M1*	Attempt at integration	
			$= \frac{1}{2}\left[4e^x\right]_0^2 \left(= 2e^2 - 2\right)$	A1		Limits not required for M and A marks
			$\bar{y} = \frac{2e^2 - 2}{4(e-1)} = \frac{e+1}{2}$	M1dep* A1 **[6]**	M1 for $\bar{y} = \frac{A\bar{y}}{A}$	
		(ii)	$A_\Delta = \frac{1}{2}(3)(2e)$	B1		Or by integration
			$A_2\bar{x} = \int_0^2 x\left(2e^{\frac{1}{2}x}\right)dx$	M1*	Integration by parts	Clear indication of integrating exponential terms and differentiating x term
			$= \left[4xe^{\frac{1}{2}x}\right]_0^2 - \int_0^2 4e^{\frac{1}{2}x} dx$			
			$= \left[4xe^{\frac{1}{2}x} - 8e^{\frac{1}{2}x}\right]_0^2 \; (= 8)$	A1 A1	All terms integrated correctly (A1 for one error)	
			$(A_1 + A_\Delta)\bar{x} = \text{cv}(8) + 3A_\Delta$	M1dep*	M1 for table of values idea	
			$\bar{x}\left(4e - 4 + 3e\right) = 8 + 3(3e)$	A1	Two terms correct	
			$\bar{x} = \frac{9e+8}{7e-4}$	A1 **[7]**	oe	

Question			Answer	Marks	Guidance	
5		(i)	Mass per unit length is $\frac{2m}{4a}$	B1		
			$I=\sum\frac{m}{2a}x^2\delta x=\frac{m}{2a}\int x^2\mathrm{d}x$	M1	M1 for $\int x^2\mathrm{d}x$	Limits not required for M mark
			$=\frac{m}{2a}\int_0^{4a}x^2\mathrm{d}x=\frac{m}{2a}\left[\frac{x^3}{3}\right]_0^{4a}$	A1	A1 for correct integration with limits	
			$=\frac{32}{3}ma^2$	A1	**AG** Correctly shown	
				[4]		
		(ii)	Angular momentum of particle before impact = $m(4av)$	B1		SC: If M0 then B1 for correct M of I
			Angular momentum after impact $\left(\frac{32}{3}ma^2+m(4a)^2\right)\omega$	B1		
			By conservation of angular momentum			
			$4mav=\frac{80}{3}ma^2\omega$	M1		
			$\omega=\frac{3v}{20a}$	A1		$0.15va^{-1}$
				[4]		
		(iii)	By conservation of energy	M1	Correct number of terms	SC: If M0 then B1 for each part
			$\frac{1}{2}\left(\frac{80}{3}ma^2\right)\left[\dot{\theta}^2-\left(\frac{3v}{20a}\right)^2\right]=\ldots$	A1ft	Kinetic energy terms using their I and ω	$\frac{40}{3}ma^2\dot{\theta}^2-\frac{3}{10}mv^2=\ldots$
			$\ldots=8mga(\cos\theta-1)$	A1	Potential energy terms	
			$\dot{\theta}^2=\frac{3g}{5a}(\cos\theta-1)+\frac{9v^2}{400a^2}$	A1	$k=\frac{3}{5}$	$\dot{\theta}^2=\omega^2+\frac{16mga(\cos\theta-1)}{I}$
				[4]		

Question			Answer	Marks	Guidance	
		(iv)	$\frac{3g}{5a}(-2)+\frac{9v^2}{400a^2}>0$	M1	Setting $\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2>0$ when $\theta=\pi$	Condone for the M mark = or ≥
			$v^2>\frac{160}{3}ga$	A1		
				[2]		
		(v)	$2\dot{\theta}\ddot{\theta}=\frac{3g}{5a}(-\sin\theta)\dot{\theta}$	M1	Differentiating $\dot{\theta}$ with respect to t	$-2mg(2a)-mg(4a)=\frac{80}{3}ma^2\ddot{\theta}$
			$\ddot{\theta}=-\frac{3g}{10a}$	A1	allow $\frac{3g}{10a}$	
			$R-3mg=3m\left(\frac{8}{3}a\ddot{\theta}\right)$	M1	For transverse acceleration $r\alpha$ - mass must be $3m$	Allow $r=a$ for the M mark
			$R=\frac{3}{5}mg$	A1		
				[4]		

Question			Answer	Marks	Guidance	
6		(i)	$I_{particle} = x^2$	B1		
			$I_{rod} = \frac{1}{3}(3)\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2} - x\right)^2$	M1 A1	M1 for correct M of I about the centre of the rod or correct application of parallel axis theorem	$I_{rod} = \frac{1}{4} + \frac{3}{4} - 3x + 3x^2$
			$I_{disc} = \frac{1}{2}(6)\left(\frac{1}{3}\right)^2 + 6\left(\frac{1}{3} + 1 - x\right)^2$	M1 A1	M1 for correct M of I about the centre of the disc or correct application of parallel axis theorem	$I_{disc} = \frac{1}{3} + \frac{32}{3} - 16x + 6x^2$
			$I_{particle} + I_{rod} + I_{disc} = 10x^2 - 19x + 12$	A1 **[6]**	**AG** Correctly shown	
		(ii)	C of M of pendulum (from A): $3\left(\frac{1}{2}\right) + 6\left(\frac{4}{3}\right) = 10\bar{x}$	M1	C of M about any point on the pendulum	Or moment of all three weights separately (M1 A1 - lhs below correct (without $g\sin\theta$)) $\left(x - 3\left(\frac{1}{2} - x\right) - 6\left(\frac{4}{3} - x\right)\right)g\sin\theta$ $= \left(10x^2 - 19x + 12\right)\ddot{\theta}$
			C of M from P: $= \frac{19}{20} - x$	A1		
			$\left(10x^2 - 19x + 12\right)\ddot{\theta} = -10g\left(\frac{19}{20} - x\right)\sin\theta$	M1	Applying $C = I\ddot{\theta}$ with their C of M	
			$\ddot{\theta} = -\frac{g}{2}\left(\frac{19 - 20x}{10x^2 - 19x + 12}\right)\sin\theta$	A1 **[4]**		Or with small angle approx. $\ddot{\theta} = -\frac{g}{2}\left(\frac{19 - 20x}{10x^2 - 19x + 12}\right)\theta$

Question			Answer	Marks	Guidance	
		OR	$E = xg\cos\theta - 3g\left(\frac{1}{2}-x\right)\cos\theta - 6g\left(\frac{4}{3}-x\right)\cos\theta + \ldots$	M1	$E = T + V$ (4 terms)	
			$\ldots + \frac{1}{2}\left(10x^2-19x+12\right)\dot{\theta}^2$	A1	Cao	
			$\left(10x^2-19x+12\right)\ddot{\theta} + \left(\frac{19}{2}-10x\right)g\sin\theta = 0$	M1 A1	M1 for differentiating their energy equation	
		(iii)	For small $\theta, \sin\theta \approx \theta$ $\ddot{\theta} = -\frac{g}{2}\left(\frac{19-20x}{10x^2-19x+12}\right)\theta$	M1	Apply small angle approximation and use of $T = \frac{2\pi}{\omega}$	
			$T = 2\pi\sqrt{\frac{20x^2-38x+24}{g(19-20x)}}$	A1 **[2]**	**AG** Clearly shown – must state that the motion is (approx.) simple harmonic	
		(iv)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{20x^2-38x+24}{19-20x}\right) = 0$	M1	Clear attempt to differentiate $\frac{20x^2-38x+24}{19-20x}$ oe and putting this expression (or just numerator) equal to zero	
			$200x^2-380x+121=0$	A1	For a correct 3 term quadratic	
			$x = 0.405$ (3 sf)	A1 **[3]**	Only (not 1.4954…)	$x = 0.40456\ldots$ or $\frac{19-\sqrt{119}}{20}$