OCR

Tuesday 24 June 2014 – Morning

A2 GCE MATHEMATICS

4731/01 Mechanics 4

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4731/01
- List of Formulae (MF1)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

* 3 1 6 9 5 6 9 5 4 2 *

INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.
- The Question Paper will be found inside the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book**. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION FOR CANDIDATES

- This information is the same on the Printed Answer Book and the Question Paper.
- The number of marks is given in brackets **[ ]** at the end of each question or part question on the Question Paper.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **12** pages. The Question Paper consists of **8** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

- Do not send this Question Paper for marking; it should be retained in the centre or recycled. Please contact OCR Copyright should you wish to re-use this document.

1 Alan is running in a straight line on a bearing of 090° at a constant speed of $4\,\mathrm{m\,s^{-1}}$. Ben sees Alan when they are 50 m apart and Alan is on a bearing of 060° from Ben. Ben sets off immediately to intercept Alan by running at a constant speed of $6\,\mathrm{m\,s^{-1}}$.

(i) Calculate the bearing on which Ben should run to intercept Alan. **[3]**

(ii) Calculate the magnitude of the velocity of Ben relative to Alan and find the time it takes, from the moment Ben sees Alan, for Ben to intercept Alan. **[4]**

2 A uniform solid circular cone has mass M and base radius R.

(i) Show by integration that the moment of inertia of the cone about its axis of symmetry is $\frac{3}{10}MR^2$. (You may assume the standard formula $\frac{1}{2}mr^2$ for the moment of inertia of a uniform disc about its axis and that the volume of a cone is $\frac{1}{3}\pi r^2 h$.) **[6]**

The axis of symmetry of the cone is fixed vertically and the cone is rotating about its axis at an angular speed of $6\,\mathrm{rad\,s^{-1}}$. A frictional couple of constant moment 0.027 N m is applied to the cone bringing it to rest. Given that the mass of the cone is 2 kg and its base radius is 0.3 m, find

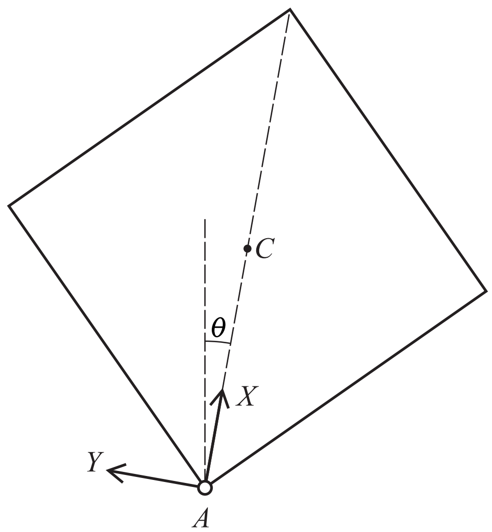
(ii) the constant angular deceleration of the cone, **[3]**

(iii) the time taken for the cone to come to rest from the instant that the couple is applied. **[2]**

3 The region bounded by the y-axis and the curves $y = \sin 2x$ and $y = \sqrt{2}\cos x$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$ is occupied by a uniform lamina. Find the exact value of the x-coordinate of the centre of mass of the lamina. **[8]**

4 A uniform square lamina has mass m and sides of length $2a$.

(i) Calculate the moment of inertia of the lamina about an axis through one of its corners perpendicular to its plane. **[3]**



The uniform square lamina has centre C and is free to rotate in a vertical plane about a fixed horizontal axis passing through one of its corners A. The lamina is initially held such that AC is vertical with C above A. The lamina is slightly disturbed from rest from this initial position. When AC makes an angle θ with the upward vertical, the force exerted by the axis on the lamina has components X parallel to AC and Y perpendicular to AC (see diagram).

(ii) Show that the angular speed, ω, of the lamina satisfies $a\omega^2 = \frac{3}{4}g\sqrt{2}(1-\cos\theta)$. **[4]**

(iii) Find X and Y in terms of m, g and θ. **[6]**

Question 5 begins on page 4.

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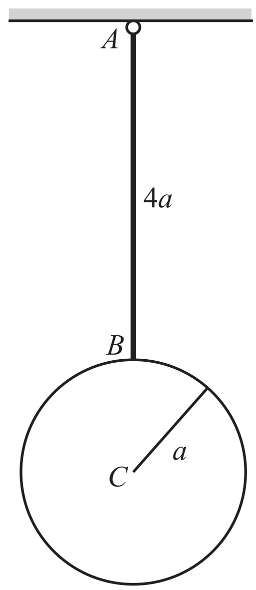


A pendulum consists of a uniform rod AB of length $4a$ and mass $4m$ and a spherical shell of radius a, mass m and centre C. The end B of the rod is rigidly attached to a point on the surface of the shell in such a way that ABC is a straight line. The pendulum is initially at rest with B vertically below A and it is free to rotate in a vertical plane about a smooth fixed horizontal axis passing through A (see diagram).

(i) Show that the moment of inertia of the pendulum about the axis of rotation is $47ma^2$. **[4]**

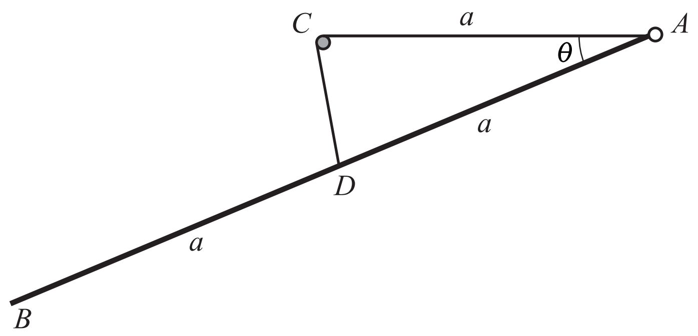
A particle of mass m is moving horizontally in the plane in which the pendulum is free to rotate. The particle has speed $\sqrt{kga}$, where k is a positive constant, and strikes the rod at a distance $3a$ from A. In the subsequent motion the particle adheres to the rod and the combined rigid body P starts to rotate.

(ii) Show that the initial angular speed of P is $\frac{3}{56}\sqrt{\frac{kg}{a}}$. **[4]**

(iii) For the case $k = 4$, find the angle that P has turned through when P first comes to instantaneous rest. **[4]**

(iv) Find the least value of k such that the rod reaches the horizontal. **[2]**

6



A uniform rod AB has mass m and length $2a$. The rod can rotate in a vertical plane about a smooth fixed horizontal axis passing through A. One end of a light elastic string of natural length a and modulus of elasticity $\sqrt{3}mg$ is attached to A. The string passes over a small smooth fixed pulley C, where AC is horizontal and $AC = a$. The other end of the string is attached to the rod at its mid-point D. The rod makes an angle θ below the horizontal (see diagram).

(i) Taking A as the reference level for gravitational potential energy, show that the total potential energy V of the system is given by

$$V = mga(\sqrt{3} - \sin\theta - \sqrt{3}\cos\theta).$$ **[4]**

(ii) Show that $\theta = \frac{1}{6}\pi$ is a position of stable equilibrium for the system. **[5]**

The system is making small oscillations about the equilibrium position.

(iii) By differentiating the energy equation with respect to time, show that

$$\frac{4}{3}a\ddot{\theta} = g(\cos\theta - \sqrt{3}\sin\theta).$$ **[4]**

(iv) Using the substitution $\theta = \phi + \frac{1}{6}\pi$, show that the motion is approximately simple harmonic, and find the approximate period of the oscillations. **[6]**

END OF QUESTION PAPER

Question			Answer	Marks	Guidance	
1		(i)		B1	Correct velocity triangle	**Or** $(6\sin\beta)t = 4t + 50\sin 60$ **and** $(6\cos\beta)t = 50\cos 60$ (or in terms of α only)
			$\frac{6}{\sin 150} = \frac{4}{\sin\alpha}$	M1	Implies previous B1 sine rule leading to $\alpha = \ldots$ (allow this mark for $\sin 30$)	Genuine attempt to solve for M1
			Bearing is $\beta = \alpha + 60° = 079.5°$	A1		79.471220…
				[3]		
		(ii)	$\frac{6}{\sin 150} = \frac{w}{\sin(30 - 19.47\ldots)}$	M1*	Use of sine rule with $30 -$ their α (or $150 -$ theirα)	Or cosine rule $w^2 = 4^2 + 6^2 - 2(4)(6)\cos(30-\alpha)$ Or $\sqrt{(6\sin\beta - 4)^2 + (6\cos\beta)^2}$
			$w = 2.19$	A1		2.1927526…
				M1 dep*	M1 for use of $s = ut$ with their w	
			$t = \frac{50}{w} = 22.8$	A1		22.802389…
				[4]		

Question			Answer	Marks	Guidance	
2		(i)	Mass per unit volume is $\rho = \dfrac{M}{\frac{1}{3}\pi R^2 h}$	B1		
			$I = \sum \frac{1}{2}(\rho\pi y^2 \delta x)y^2 = \frac{1}{2}\rho\pi\int y^4 \mathrm{d}x$	M1	M1 for $\cdots\int y^4 \mathrm{d}x$	
			$= \frac{1}{2}\rho\pi\int_0^h \frac{R^4x^4}{h^4}\mathrm{d}x$	M1A1	M1 for substituting $y = \frac{R}{h}x$ A1 for correct integral with limits	
			$= \frac{1}{2}\rho\pi\left[\frac{R^4x^5}{5h^4}\right]_0^h$	A1	A1 for correct integration	
			$= \frac{1}{10}\left(\frac{3M}{\pi R^2 h}\right)\pi R^4 h = \frac{3}{10}MR^2$	A1	**AG** Correctly shown	
				[6]		
		(ii)	MI of cone $= \frac{3}{10}(2)(0.3)^2$	B1	Use of $\frac{3}{10}mr^2$ with $m = 2$ and $r = 0.3$	
			$0.027 = 0.054\alpha$	M1	Using $C = I\alpha$	
			$\alpha = 0.5$ rad s^{-2}	A1		
				[3]		
		(iii)	$6 - 0.5t = 0$	M1	Use of $\omega = \omega_0 + \alpha t$	
			$t = 12s$	A1	Cao	
				[2]		

Question			Answer	Marks	Guidance	
3			$A=\int_0^{\frac{\pi}{4}}\left(\sqrt{2}\cos x-\sin 2x\right)\mathrm{d}x$	M1*	Attempt at integration to find area (both terms including subtraction)	Limits not required for M and first A mark.
			$=\left[\sqrt{2}\sin x+\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{4}}=\frac{1}{2}$	A1A1	A1 for both terms correct, A1 for $\frac{1}{2}$	
			$A\bar{x}=\int_0^{\frac{\pi}{4}}x\left(\sqrt{2}\cos x-\sin 2x\right)\mathrm{d}x$	M1*	Integration by parts	Clear indication of integrating trigonometric term and differentiating x term
			$=\left[x\left(\sqrt{2}\sin x+\frac{1}{2}\cos 2x\right)\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\left(\sqrt{2}\sin x+\frac{1}{2}\cos 2x\right)\mathrm{d}x$			
			$=\left[x\left(\sqrt{2}\sin x+\frac{1}{2}\cos 2x\right)+\sqrt{2}\cos x-\frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{4}}$ $=\frac{\pi}{4}+\frac{3}{4}-\sqrt{2}$	A2	Both terms integrated correctly (A1 for one error)	Limits not required for M and A marks (for $A\bar{x}$)
			$\bar{x}=\frac{\left(\frac{\pi}{4}+\frac{3}{4}-\sqrt{2}\right)}{\left(\frac{1}{2}\right)}\quad=\frac{\pi}{2}+\frac{3}{2}-2\sqrt{2}$	M1 dep* A1	M1 for $\bar{x}=\frac{A\bar{x}}{A}$	
				[8]		

Question			Answer	Marks	Guidance	
		OR	$\dfrac{\int_0^{\frac{\pi}{4}} x\sin 2x\,dx}{\int_0^{\frac{\pi}{4}} \sin 2x\,dx} = \dfrac{\left[-\frac{x}{2}\cos 2x\right]_0^{\frac{\pi}{4}} + \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos 2x\,dx}{\left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{4}}}$	M1* A1	Attempt at integration to find the centre of mass for the lamina bounded by $y = \sin 2x$ A1 for correct first stage of parts in numerator and correct denominator (ignore limits for this mark)	For the first 2 M marks must be a genuine attempt at integrating by parts
			$= \dfrac{\left[-\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{4}}}{\frac{1}{2}} = \dfrac{1}{2}$	A1	Cao	
			$\dfrac{\int_0^{\frac{\pi}{4}} x\sqrt{2}\cos x\,dx}{\int_0^{\frac{\pi}{4}} \sqrt{2}\cos x\,dx} = \dfrac{\sqrt{2}\left[x\sin x\right]_0^{\frac{\pi}{4}} - \sqrt{2}\int_0^{\frac{\pi}{4}} \sin x\,dx}{\left[\sqrt{2}\sin x\right]_0^{\frac{\pi}{4}}}$	M1* A1	Attempt at integration to find the centre of mass for the lamina bounded by $y = \sqrt{2}\cos x$ A1 for correct first stage of parts in numerator and correct denominator (ignore limits for this mark)	
			$= \dfrac{\sqrt{2}\left[x\sin x + \cos x\right]_0^{\frac{\pi}{4}}}{1} = \sqrt{2}\left(\dfrac{\pi\sqrt{2}}{8} + \dfrac{\sqrt{2}}{2} - 1\right)$	A1	Cao	
			$\left(1 - \dfrac{1}{2}\right)\bar{x} = \left(\dfrac{\pi}{4} + 1 - \sqrt{2}\right)(1) - \dfrac{1}{2}(\dfrac{1}{2})$	M1 dep*	Taking moments – signs must be consistent	
			$\bar{x} = \dfrac{\pi}{2} + \dfrac{3}{2} - 2\sqrt{2}$	A1		

Question			Answer	Marks	Guidance	
4		(i)	MI square centre $I_c = \frac{1}{3}m\left(a^2 + a^2\right)$ $I_A = \frac{1}{3}m\left(2a^2\right) + m\left(a\sqrt{2}\right)^2 = \frac{8}{3}ma^2$	B1 M1A1 **[3]**	May be implied by later working M1 for applying parallel axes rule	B1 $I_x = \frac{4}{3}ma^2$ M1 applying perpendicular axes rule ($I_x = I_y$)
		(ii)	$\frac{1}{2}I\omega^2 = mga\sqrt{2}\left(1-\cos\theta\right)$ $\frac{1}{2}\left(\frac{8}{3}ma^2\right)\omega^2 = mga\sqrt{2}(1-\cos\theta)$ $a\omega^2 = \frac{3}{4}g\sqrt{2}(1-\cos\theta)$	M1 A1 A1 A1 **[4]**	Equation involving KE and PE A1 for KE term, A1 for PE term **AG** Correctly obtained	
		(iii)	$2a\omega\frac{d\omega}{dt} = \frac{3g\sqrt{2}}{4}\sin\theta\frac{d\theta}{dt}$ $\alpha = \frac{3g\sqrt{2}}{8a}\sin\theta$ $mg\cos\theta - X = ma\sqrt{2}\omega^2$ $X = \frac{1}{2}mg(5\cos\theta - 3)$ $mg\sin\theta - Y = ma\sqrt{2}\alpha$ $Y = \frac{1}{4}mg\sin\theta$	M1 A1 M1 A1 M1 A1 **[6]**	Differentiating ω with respect to t For radial acceleration $r\omega^2$ For transverse acceleration $r\alpha$	Or M1 for applying $C = I\alpha$ with their I from (i) A1 for $a\sqrt{2}mg\sin\theta = \frac{8}{3}ma^2\alpha$

Question			Answer	Marks	Guidance	
5		(i)	$I_{rod} = \frac{4}{3}(4m)(2a)^2 \left(= \frac{64}{3}ma^2\right)$	B1		
			$I_{shell} = \frac{2}{3}ma^2 + m(5a)^2$	M1A1	Attempt at MI of shell about the axis A using the parallel axes rule	
			$I = \frac{66}{3}ma^2 + 25ma^2 = 47ma^2$	A1 **[4]**	**AG** Correctly shown	
		(ii)	Angular momentum of particle before impact $= 3a\left(m\sqrt{kga}\right)$	B1		
			Angular momentum after impact $= \left(47ma^2 + m(3a)^2\right)\omega$	B1		
			By conservation of angular momentum $3ma\sqrt{kga} = 56ma^2\omega$	M1		
			$\omega = \frac{3}{56}\sqrt{\frac{kg}{a}}$	A1 **[4]**	**AG** Correctly shown	
		(iii)	By conservation of energy $-16mga + \frac{1}{2}\left(56ma^2\right)\left(\frac{3}{56}\sqrt{\frac{kg}{a}}\right)^2 = 0 - 16mga\cos\theta$	B1* B1	B1* for kinetic energy, B1 for potential energy	
			$16\cos\theta = 16 - 112\left(\frac{3}{56}\right)^2$	M1 dep*	Conservation of energy – must be using correct mass and V must of the correct form	
			$\theta = 0.201$	A1 **[4]**	0.2007830…	Degrees: 11.504019...
		(iv)	$16mga = \frac{1}{2}\left(56ma^2\right)\omega^2$	M1	KE loss = PE gain	Must be using correct mass of $6m$
			$k = 199$	A1 **[2]**	Cao (accept $k = 200$)	$k = \frac{1792}{9} = 199.111...$

Question			Answer	Marks	Guidance	
6		(i)	$GPE = -mga\sin\theta$	B1		
			$x^2 = a^2 + a^2 - 2(a)(a)\cos\theta$	B1	Or $x = 2a\sin\left(\frac{\theta}{2}\right)$	
			$EPE = \frac{\sqrt{3}mg\{2a^2(1-\cos\theta)\}}{2a}$	M1	Using $\frac{\lambda x^2}{2a}$	Genuine attempt at extension
			$V = mga(\sqrt{3} - \sin\theta - \sqrt{3}\cos\theta)$	A1	**AG** Correctly shown	
				[4]		
		(ii)	$\frac{dV}{d\theta} = mga(-\cos\theta + \sqrt{3}\sin\theta) = 0$	M1	Attempt at differentiation	
				A1	Correct derivative and equal to zero	
			$\sqrt{3}\tan\theta - 1 = 0 \Rightarrow \theta = \frac{\pi}{6}$	A1	**AG** Correctly shown	Accept substitution of $\theta = \frac{\pi}{6}$ into V' to show that $V' = 0$ for second A mark
			$\frac{d^2V}{d\theta^2} = mga\left(\sin\theta + \sqrt{3}\cos\theta\right)$	M1	Attempt at second derivative (or first derivative) test	
			when $\theta = \frac{\pi}{6}, \frac{d^2V}{d\theta^2} = 2mga > 0$, so equilibrium is stable	A1	Correctly value of V'' and > 0	
				[5]		
		(iii)	KE of system: $T = \frac{1}{2}\left(\frac{4}{3}ma^2\right)\dot{\theta}^2$	B1		
			$\frac{2}{3}ma^2\dot{\theta}^2 + mga\left(\sqrt{3} - \sin\theta - \sqrt{3}\cos\theta\right) = E$	M1	Attempt at formulation of relevant energy equation (T + V = E)	
			$\frac{4}{3}ma^2\dot{\theta}\ddot{\theta} + mga\left(-\cos\theta\dot{\theta} + \sqrt{3}\sin\theta\dot{\theta}\right) = 0$	M1	Differentiates their energy equation	Condone absence of $\dot{\theta}$ throughout – but if inconsistent then M0
			$\frac{4}{3}a\ddot{\theta} = g\left(\cos\theta - \sqrt{3}\sin\theta\right)$	A1	**AG** Correctly shown	
				[4]		

Question			Answer	Marks	Guidance	
		(iv)	$\frac{4}{3}a\ddot{\phi} = g\cos\left(\phi + \frac{\pi}{6}\right) - \sqrt{3}g\sin\left(\phi + \frac{\pi}{6}\right)$	M1*	Substituting $\theta = \phi + \frac{\pi}{6}$ and genuine attempt to expand both terms	
			$\frac{4}{3}a\ddot{\phi} = g\left(\cos\phi\cos\frac{\pi}{6} - \sin\phi\sin\frac{\pi}{6}\right)$ $-\sqrt{3}g\left(\sin\phi\cos\frac{\pi}{6} + \sin\frac{\pi}{6}\cos\phi\right)$			
			$\frac{4}{3}a\ddot{\phi} = g\left(\frac{\sqrt{3}}{2}\cos\phi - \frac{1}{2}\sin\phi\right) - \sqrt{3}g\left(\frac{\sqrt{3}}{2}\sin\phi + \frac{1}{2}\cos\phi\right)$	A2	Correct trigonometric expansion, $\ddot{\theta} = \ddot{\phi}$ and using both $\sin\frac{\pi}{6} = \frac{1}{2}$, $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ (one error A1)	
			$\frac{4}{3}a\ddot{\phi} = -2g\sin\phi$			
			For small $\phi, \sin\phi \approx \phi$	M1 dep*	Apply small angle approximation	
			$\ddot{\phi} \approx -\frac{3g}{2a}\phi$, SHM	A1	$\ddot{\phi} \approx -\omega^2\phi$ and must state this is (approx.) simple harmonic – condone $\ddot{\theta}$	
			Approximate period is $2\pi\sqrt{\frac{2a}{3g}}$	A1	Cao	
				[6]		