OCR RECOGNISING ACHIEVEMENT

Monday 10 June 2013 – Morning

A2 GCE MATHEMATICS

4727/01 Further Pure Mathematics 3

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4727/01
- List of Formulae (MF1)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

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INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.

- The Question Paper will be found in the centre of the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book**. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.

INFORMATION FOR CANDIDATES

This information is the same on the Printed Answer Book and the Question Paper.

- The number of marks is given in brackets **[ ]** at the end of each question or part question on the Question Paper.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **16** pages. The Question Paper consists of **4** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

- Do not send this Question Paper for marking; it should be retained in the centre or recycled. Please contact OCR Copyright should you wish to re-use this document.

1 The plane Π passes through the points with coordinates (1, 6, 2), (5, 2, 1) and (1, 0, –2).

(i) Find a vector equation of Π in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. **[2]**

(ii) Find a cartesian equation of Π. **[4]**

2 G consists of the set $\{1, 3, 5, 7\}$ with the operation of multiplication modulo 8.

(i) Write down the operation table and, assuming associativity, show that G is a group. **[5]**

(ii) State the order of each element. **[1]**

(iii) Find all the proper subgroups of G. **[1]**

The group H consists of the set $\{1, 3, 7, 9\}$ with the operation of multiplication modulo 10.

(iv) Explaining your reasoning, determine whether H is isomorphic to G. **[2]**

3 The differential equation

$$3xy^2\frac{dy}{dx} + 2y^3 = \frac{\cos x}{x}$$

is to be solved for $x > 0$. Use the substitution $u = y^3$ to find the general solution for y in terms of x. **[8]**

4 The complex numbers 0, 3 and $3e^{\frac{1}{3}\pi i}$ are represented in an Argand diagram by the points O, A and B respectively.

(i) Sketch the triangle OAB and show that it is equilateral. **[3]**

(ii) Hence express $3 - 3e^{\frac{1}{3}\pi i}$ in polar form. **[2]**

(iii) Hence find $(3 - 3e^{\frac{1}{3}\pi i})^5$, giving your answer in the form $a + b\sqrt{3}\,i$ where a and b are rational numbers. **[3]**

5 Find the solution of the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = e^{-x}$ for which $y = \frac{dy}{dx} = 0$ when $x = 0$. **[11]**

6 The plane Π has equation $x + 2y - 2z = 5$. The line l has equation $\frac{x-1}{2} = \frac{y+1}{5} = \frac{z-2}{1}$.

(i) Find the coordinates of the point of intersection of l with the plane Π. **[3]**

(ii) Calculate the acute angle between l and Π. **[3]**

(iii) Find the coordinates of the two points on the line l such that the distance of each point from the plane Π is 2. **[5]**

7 A commutative group G has order 18. The elements a, b and c have orders 2, 3 and 9 respectively.

(i) Prove that ab has order 6. **[4]**

(ii) Show that G is cyclic. **[3]**

8 **(i)** Use de Moivre's theorem to show that $\cos 5\theta \equiv 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. **[5]**

(ii) Hence find the roots of $16x^4 - 20x^2 + 5 = 0$ in the form $\cos\alpha$ where $0 \leqslant \alpha \leqslant \pi$. **[4]**

(iii) Hence find the exact value of $\cos\frac{1}{10}\pi$. **[3]**

Question			Answer	Marks	Guidance	
1	(i)		vectors in plane: two of $\begin{pmatrix}-4\\4\\1\end{pmatrix}, \begin{pmatrix}0\\6\\4\end{pmatrix} = 2\begin{pmatrix}0\\3\\2\end{pmatrix}, \begin{pmatrix}4\\2\\3\end{pmatrix}$	M1	Differences between two pairs	Any multiple
			$\mathbf{r} = \begin{pmatrix}1\\6\\2\end{pmatrix} + \lambda\begin{pmatrix}0\\3\\2\end{pmatrix} + \mu\begin{pmatrix}4\\2\\3\end{pmatrix}$	A1	Aef of parametric equation	Must have "**r = ...**"
				[2]		
1	(ii)		$\begin{pmatrix}0\\3\\2\end{pmatrix} \times \begin{pmatrix}4\\2\\3\end{pmatrix} = \begin{pmatrix}5\\8\\-12\end{pmatrix}$	M1 A1	Calculate vector product or multiple	M1 can be awarded where vector product has method shown or only one term wrong
			$\left(\mathbf{r} - \begin{pmatrix}1\\6\\2\end{pmatrix}\right) \cdot \begin{pmatrix}5\\8\\-12\end{pmatrix} = 0$	M1		Or Cartesian form = d with attempt to compute d
			$5x + 8y - 12z = 29$	A1	Aef of cartesian equation, isw.	
				[4]		
			Alternate method			
				M1 A1 M1A1	EITHER x, y, z in parametric form both parameters in terms of e.g. x, y substitute into parametric form of z	
				M1 A1 M1 A1	OR x, y, z in parametric form 2 equations in x, y, z and one parameter eliminate parameter	

Question			Answer	Marks	Guidance	
2	(i)		$\begin{array}{c\|cccc} & 1 & 3 & 5 & 7 \\ \hline 1 & 1 & 3 & 5 & 7 \\ 3 & 3 & 1 & 7 & 5 \\ 5 & 5 & 7 & 1 & 3 \\ 7 & 7 & 5 & 3 & 1 \end{array}$	B2	–1 each error	
			From table clearly closed	B1		Must be clear they are referring to tabulated results
			1 is identity	B1		
			$3^{-1} \equiv 3,\ 5^{-1} \equiv 5,\ 7^{-1} \equiv 7 \pmod 8$	B1		Or "1 appears in every row"
				[5]	Superfluous fact/s gets –1	
2	(ii)		1 has order 1 and 3, 5, 7 all have order 2	B1 **[1]**		
2	(iii)		{1, 3}, {1, 5}, {1, 7} (and {1})	B1 **[1]**	All correct, no extras	Allow {1} included or omitted
2	(iv)		in H $3^2 \equiv 9 \pmod{10}$ so 3 not order 2	M1	Shows and states that 3 or that 7 is not order 2 (or **is** order 4)	
			no element of order > 2 in G so not isomorphic	A1 **[2]**	Completely correct reasoning **Or**, if zero, then **SC1** for merely stating comparable orders and then saying that "orders don't correspond, so not isomorphic" **Or** table for H with saying "not all elements self inverse, so not isomorphic"	

Question			Answer	Marks	Guidance	
3			$u = y^3 \Rightarrow \frac{du}{dx} = 3y^2 \frac{dy}{dx}$	M1		Or $\frac{dy}{dx} = \frac{1}{3}u^{-\frac{2}{3}}\frac{du}{dx}$
			in DE gives $x\frac{du}{dx} + 2u = \frac{\cos x}{x}$	A1		
			$\frac{du}{dx} + \frac{2}{x}u = \frac{\cos x}{x^2}$	B1	Divide	Both sides
			$I = \exp\left(\int \frac{2}{x}dx\right) = e^{2\ln x}$	M1	Correctly integrates	Must have form $\frac{du}{dx} + f(x)u = g(x)$
			$= x^2$	A1		Can be implied by subsequent work
			$x^2\frac{du}{dx} + 2xu = \cos x$			
			$\frac{d}{dx}\left(x^2 u\right) = \cos x$			
			$x^2 u = \sin x \quad (+A)$	M1	Integrate	
			$u = \frac{\sin x + A}{x^2}$	A1	Or gives GS in implicit form	Must include constant at this stage
			$y = \left(\frac{\sin x + A}{x^2}\right)^{\frac{1}{3}}$	A1		
				[8]		

Question			Answer	Marks	Guidance	
4	(i)		Sketch	B1		Must have axes, A shown 3 across and either scale (or co-ordinates) with B in rough position, or angle and distance on argand diagram. No inconsistencies
			$OA = \lvert 3\rvert = 3,\ OB = \left\lvert 3\mathrm{e}^{\frac{1}{3}\pi \mathrm{i}}\right\rvert = 3$			
			and $\angle BOA = \frac{1}{3}\pi$	M1	Can be seen on diagram	Alt. Attempts AB or second angle
			hence ΔOAB equilateral	A1		
				[3]		
4	(ii)		$3\mathrm{e}^{-\frac{1}{3}\pi \mathrm{i}}$	M1A1	Or $3\mathrm{e}^{\frac{5}{3}\pi \mathrm{i}}$. Isw M1 for evidence they are considering BA, or for $\frac{3}{2} - \frac{3}{2}\sqrt{3}\,\mathrm{i}$	For full marks can use CiS form, or clear polar co-ordinates, in radians. Not C-iS
				[2]		
4	(iii)		$\left(3 - 3\mathrm{e}^{\frac{1}{3}\pi \mathrm{i}}\right)^5 = 3^5\,\mathrm{e}^{-\frac{5}{3}\pi \mathrm{i}}$	M1	For mod^5 and arg × 5	"Hence" so must use 'their $3\mathrm{e}^{-\frac{1}{3}\pi \mathrm{i}}$'
			$= 243\left(\cos\frac{5}{3}\pi - \mathrm{i}\sin\frac{5}{3}\pi\right)$	A1ft	aef	
			$= \frac{243}{2} + \frac{243}{2}\sqrt{3}\,\mathrm{i}$	B1		Condone use of "121.5".
				[3]		

Question			Answer	Marks	Guidance	
5			AE: $\lambda^2 + 2\lambda + 5 = 0$	M1		
			$\lambda = -1 \pm 2\mathrm{i}$	A1		
			CF: $\mathrm{e}^{-x}\left(A\cos 2x + B\sin 2x\right)$	A1ft		Or $A\mathrm{e}^{-x}\cos(2x+\alpha)$ Must be in real form
			PI: $y = a\mathrm{e}^{-x}$	B1		If PI $y = ax\mathrm{e}^{-x}$,then max of M1,A1,A1, B0,M1,A0,A0 (since cannot be consistent) M1, M1, A1.
			$a\mathrm{e}^{-x} - 2a\mathrm{e}^{-x} + 5a\mathrm{e}^{-x} = \mathrm{e}^{-x}$ $4a = 1$	M1	Differentiate & substitute	Must have a constant in "their PI"
			$a = \frac{1}{4}$	A1		
			GS: $y = \mathrm{e}^{-x}\left(\frac{1}{4} + A\cos 2x + B\sin 2x\right)$	A1ft		Must have "$y =$"
			$\frac{\mathrm{d}y}{\mathrm{d}x} = -\mathrm{e}^{-x}\left(\frac{1}{4} + A\cos 2x + B\sin 2x\right)$ $+\mathrm{e}^{-x}\left(-2A\sin 2x + 2B\cos 2x\right)$	M1*	Differentiate their GS of form $y = \mathrm{e}^{-x}\left(P + A\cos nx + B\sin nx\right)$ where P is constant or linear term, n not 0 or 1	Allow one error
			$x = 0, \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow -\left(\frac{1}{4} + A\right) + 2B = 0$ $x = 0, y = 0 \Rightarrow \frac{1}{4} + A = 0$	*M1	Use conditions	But M0 if leads to solution of $y = 0$
			$A = -\frac{1}{4},\ B = 0$	A1ft	From their GS	
			$y = \frac{1}{4}\mathrm{e}^{-x}\left(1 - \cos 2x\right)$	A1 **[11]**		Must have '$y =$' and be in real form
6	(i)		$x = 2t+1,\ y = 5t-1,\ z = t+2$	B1	Parameterise	or B1 for y and z correctly in terms of x e.g. $2y = 5x - 7$, $2z = x + 3$ Then M1 for full simultaneous equations method.
			$(2t+1) + 2(5t-1) - 2(t+2) = 5$		Substitute into plane	
			$\Rightarrow 10t = 10 \Rightarrow t = 1$	M1	Solve	
			Intersect at (3, 4, 3)	A1 **[3]**	cao	Accept vector form

Question			Answer	Marks	Guidance	
6	(ii)		$\cos\left(\frac{1}{2}\pi-\theta\right)=\dfrac{\left|\begin{pmatrix}2\\5\\1\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-2\end{pmatrix}\right|}{\left|\begin{pmatrix}2\\5\\1\end{pmatrix}\right|\left|\begin{pmatrix}1\\2\\-2\end{pmatrix}\right|}=\dfrac{10}{3\sqrt{30}}$	M1A1		Attempt to find angle or its complement
			$\theta=0.654$	A1 [3]	or 37.5°	
6	(iii)		If P is point of intersection and Q is required point, $\overrightarrow{PQ}=\lambda\begin{pmatrix}2\\5\\1\end{pmatrix}$ so $\sin\theta=\dfrac{2}{PQ}=\dfrac{2}{\lvert\lambda\rvert\sqrt{30}}$	M1*	or $\overrightarrow{PQ}.\hat{\mathbf{n}}=\pm 2$ where $\mathbf{n}=\begin{pmatrix}1\\2\\-2\end{pmatrix}$	Use $\overrightarrow{PQ}$ with right angled triangle or consider component of $\overrightarrow{PQ}$ in direction of normal vector.
			$\dfrac{10}{3\sqrt{30}}=\dfrac{2}{\lvert\lambda\rvert\sqrt{30}}$	M1		Valid method to set up equation in λ alone.
			$\lambda=\pm\dfrac{3}{5}$	A1		(May work from general point on original equation)
			points have position vectors $\begin{pmatrix}3\\4\\3\end{pmatrix}\pm\dfrac{3}{5}\begin{pmatrix}2\\5\\1\end{pmatrix}$	*M1	Dep on 1st M1	
			points at (1.8, 1, 2.4) and (4.2, 7, 3.6)	A1	cao	
			Alternative: Distance $=\dfrac{\lvert 2t+1+2(5t-1)-2(t+2)-5\rvert}{\sqrt{1^2+2^2+2^2}}=2$	M1* A1		Zero if formula used without substitution in of parametric form.
			$\Rightarrow t=0.4$ or 1.6	*M1 A1	Solve; At least one value found	
			(1.8, 1, 2.4) and (4.2, 7, 3.6)	A1 [5]		

Question			Answer	Marks	Guidance	
7	(i)		$(ab)^6 = abab...ab = a^6b^6$ as commutative	M1	Must give reason	Some demonstration that they understand commutativity
			$= \left(a^2\right)^3\left(b^3\right)^2 = e^3e^2 = e$	A1	Using orders of a and b	
			So ab has order 1, 2, 3, or 6			
			($b \neq a \Rightarrow ab \neq a^2 \Rightarrow ab \neq e$ so ab not order 1)			Condone absence of this line
			$(ab)^2 = a^2b^2 = eb^2 = b^2$ and b not order 2, so ab not order 2	M1	Consider other cases	Insufficient to merely have simplified **all** $(ab)^n$
			$(ab)^3 = a^3b^3 = aa^2e = aee = a \neq e$, so ab not order 3			
			(So must be order 6)	A1 **[4]**	AG Complete argument	
7	(ii)		ac has order 18	B1		Or abc or generator
			18 is LCM of 2 and 9 , (so we can use a similar argument to part (i))	M1	or explicit consideration of other cases	
			So as G has an element of order 18 it must be cyclic.	A1 **[3]**	AG Complete argument	
8	(i)		$\cos 5\theta + \mathrm{i}\sin 5\theta = (\cos\theta + \mathrm{i}\sin\theta)^5$	B1	Or $\cos 5\theta = re\{(\cos\theta + \mathrm{i}\sin\theta)^5\}$	
			$= c^5 + 5\mathrm{i}c^4s - 10c^3s^2 - 10\mathrm{i}c^2s^3 + 5cs^4 + \mathrm{i}s^5$	M1		No more than 1 error, can be unsimplified
			$\cos 5\theta = c^5 - 10c^3s^2 + 5cs^4$	M1	Take real parts	
			$= c^5 - 10c^3\left(1-c^2\right) + 5c\left(1-c^2\right)^2$	M1		
			$= c^5 - 10c^3 + 10c^5 + 5c - 10c^3 + 5c^5$			
			$\cos 5\theta = 16c^5 - 20c^3 + 5c$	A1 **[5]**	AG	

Question			Answer	Marks	Guidance	
8	(ii)		Multiplying by x gives $16x^5-20x^3+5x=0$			Hence, so no marks for using quadratic at this stage.
			letting $x=\cos\alpha$ gives $\cos 5\alpha=0$	M1		
			hence $5\alpha=\frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \frac{7}{2}\pi, \frac{9}{2}\pi$	A1		
			$\alpha=\frac{1}{10}\pi, \frac{3}{10}\pi, \frac{5}{10}\pi, \frac{7}{10}\pi, \frac{9}{10}\pi$			
			$\cos\frac{5}{10}\pi=0$ which is not a root	A1		
			so roots $x=\cos\frac{1}{10}\pi, \cos\frac{3}{10}\pi, \cos\frac{7}{10}\pi, \cos\frac{9}{10}\pi$	A1		
				[4]		
8	(iii)		$16x^4-20x^2+5=0 \Leftrightarrow x^2=\dfrac{20\pm\sqrt{80}}{32}$	B1		Can be gained if seen in (ii)
			cos decreases between 0 and π so $\cos\frac{1}{10}\pi$ is greatest root	M1		
			so $\cos\frac{1}{10}\pi=\sqrt{\dfrac{20+\sqrt{80}}{32}}=\sqrt{\dfrac{5+\sqrt{5}}{8}}$	A1	Dep on full marks in (ii)	
				[3]		