OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS

4727

Further Pure Mathematics 3

Candidates answer on the Answer Booklet

OCR Supplied Materials:
- 8 page Answer Booklet
- List of Formulae (MF1)

Other Materials Required:
None

Thursday 29 January 2009
Morning

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

1 In this question G is a group of order n, where $3 \leqslant n < 8$.

(i) In each case, write down the smallest possible value of n:

(a) if G is cyclic, **[1]**

(b) if G has a proper subgroup of order 3, **[1]**

(c) if G has at least two elements of order 2. **[1]**

(ii) Another group has the same order as G, but is not isomorphic to G. Write down the possible value(s) of n. **[2]**

2 **(i)** Express $\dfrac{\sqrt{3}+\mathrm{i}}{\sqrt{3}-\mathrm{i}}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 \leqslant \theta < 2\pi$. **[3]**

(ii) Hence find the smallest positive value of n for which $\left(\dfrac{\sqrt{3}+\mathrm{i}}{\sqrt{3}-\mathrm{i}}\right)^n$ is real and positive. **[2]**

3 Two skew lines have equations

$$\frac{x}{2} = \frac{y+3}{1} = \frac{z-6}{3} \qquad \text{and} \qquad \frac{x-5}{3} = \frac{y+1}{1} = \frac{z-7}{5}.$$

(i) Find the direction of the common perpendicular to the lines. **[2]**

(ii) Find the shortest distance between the lines. **[4]**

4 Find the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 65\sin 2x.$$ **[9]**

5 The variables x and y are related by the differential equation

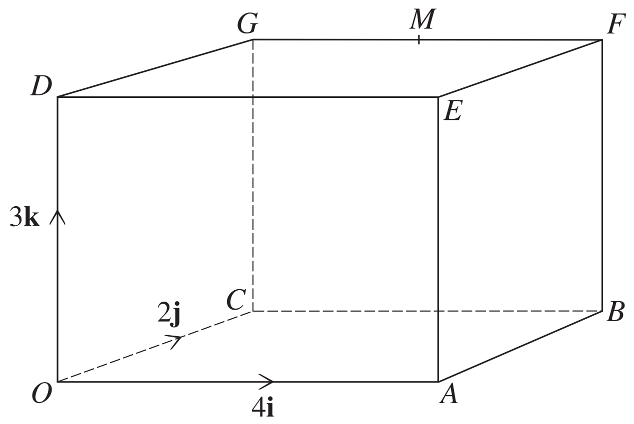
$$x^3\frac{\mathrm{d}y}{\mathrm{d}x} = xy + x + 1. \qquad \text{(A)}$$

(i) Use the substitution $y = u - \dfrac{1}{x}$, where u is a function of x, to show that the differential equation may be written as

$$x^2\frac{\mathrm{d}u}{\mathrm{d}x} = u.$$ **[4]**

(ii) Hence find the general solution of the differential equation (A), giving your answer in the form $y = \mathrm{f}(x)$. **[5]**

6



The cuboid $OABCDEFG$ shown in the diagram has $\overrightarrow{OA} = 4\mathbf{i}$, $\overrightarrow{OC} = 2\mathbf{j}$, $\overrightarrow{OD} = 3\mathbf{k}$, and M is the mid-point of GF.

(i) Find the equation of the plane $ACGE$, giving your answer in the form $\mathbf{r}.\mathbf{n} = p$. **[4]**

(ii) The plane $OEFC$ has equation $\mathbf{r}.(3\mathbf{i} - 4\mathbf{k}) = 0$. Find the acute angle between the planes $OEFC$ and $ACGE$. **[4]**

(iii) The line AM meets the plane $OEFC$ at the point W. Find the ratio $AW : WM$. **[5]**

7 **(i)** The operation $*$ is defined by $x * y = x + y - a$, where x and y are real numbers and a is a real constant.

(a) Prove that the set of real numbers, together with the operation $*$, forms a group. **[6]**

(b) State, with a reason, whether the group is commutative. **[1]**

(c) Prove that there are no elements of order 2. **[2]**

(ii) The operation $\circ$ is defined by $x \circ y = x + y - 5$, where x and y are **positive** real numbers. By giving a numerical example in each case, show that two of the basic group properties are not necessarily satisfied. **[4]**

8 **(i)** By expressing $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, show that

$$\sin^6\theta \equiv -\tfrac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10).$$ **[5]**

(ii) Replace θ by $\left(\tfrac{1}{2}\pi - \theta\right)$ in the identity in part **(i)** to obtain a similar identity for $\cos^6\theta$. **[3]**

(iii) Hence find the exact value of $\displaystyle\int_0^{\frac{1}{4}\pi} \left(\sin^6\theta - \cos^6\theta\right) d\theta$. **[4]**

4727 Further Pure Mathematics 3

1 (i) (a)	$(n=)\ 3$	B1	**1**	For correct n
(b)	$(n=)\ 6$	B1	**1**	For correct n
(c)	$(n=)\ 4$	B1	**1**	For correct n
(ii)	$(n=)\ 4,\ 6$	B1		For *either* 4 *or* 6
		B1	**2**	For both 4 and 6 and no extras Ignore all $n \geqslant 8$ **SR** B0 B0 if more than 3 values given, even if they include 4 or 6
			5	
2 (i)	$\frac{\sqrt{3}+\mathrm{i}}{\sqrt{3}-\mathrm{i}} \times \frac{\sqrt{3}+\mathrm{i}}{\sqrt{3}+\mathrm{i}} = \frac{1}{2}+\frac{1}{2}\mathrm{i}\sqrt{3}$	M1		For multiplying top and bottom by complex conjugate
	OR $\frac{\sqrt{3}+\mathrm{i}}{\sqrt{3}-\mathrm{i}} = \frac{2\mathrm{e}^{\frac{1}{6}\pi \mathrm{i}}}{2\mathrm{e}^{-\frac{1}{6}\pi \mathrm{i}}}$			*OR* for changing top and bottom to polar form
	$=(1)\mathrm{e}^{\frac{1}{3}\pi \mathrm{i}}$	A1		For $(r=)\ 1$ (may be implied)
		A1	**3**	For $(\theta =)\ \frac{1}{3}\pi$ **SR** Award maximum A1 A0 if $\mathrm{e}^{\mathrm{i}\theta}$ form is not seen
(ii)	$\left(\mathrm{e}^{\frac{1}{3}\pi \mathrm{i}}\right)^6 = \mathrm{e}^{2\pi \mathrm{i}} = 1 \Rightarrow (n=)\ 6$	M1		For use of $\mathrm{e}^{2\pi \mathrm{i}}=1$, $\mathrm{e}^{\mathrm{i}\pi}=-1$, $\sin k\pi = 0$ *or* $\cos k\pi = \pm 1$ (may be implied)
		A1	**2**	For $(n=)\ 6$ **SR** For $(n=)\ 3$ only, award M1 A0
			5	
3 (i)	$\mathbf{n} = [2, 1, 3] \times [3, 1, 5]$	M1		For using direction vectors and attempt to find vector product
	$= [2, -1, -1]$	A1	**2**	For correct direction (allow multiples)
(ii)	$d = \frac{\lvert [5, 2, 1] \cdot [2, -1, -1] \rvert}{\sqrt{6}}$	B1		For $(\mathbf{AB} =)\ [5, 2, 1]$ or any vector joining lines
		M1		For attempt at evaluating $\mathbf{AB} \cdot \mathbf{n}$
		M1		For $\lvert \mathbf{n} \rvert$ in denominator
	$= \frac{7}{\sqrt{6}} = \frac{7}{6}\sqrt{6} = 2.8577$	A1	**4**	For correct distance
			6	

4	$m^2+4m+5\ (=0) \Rightarrow m=\dfrac{-4\pm\sqrt{16-20}}{2}$	M1	For attempt to solve correct auxiliary equation
	$=-2\pm \text{i}$	A1	For correct roots
	CF $=\text{e}^{-2x}(C\cos x+D\sin x)$	A1√	For correct CF (here or later). f.t. from m **AEtrig** but not forms including $\text{e}^{\text{i}x}$
	PI $=p\sin 2x+q\cos 2x$	B1	For stating a trial PI of the correct form
	$y'=2p\cos 2x-2q\sin 2x$ $y''=-4p\sin 2x-4q\cos 2x$	M1	For differentiating PI twice and substituting into the DE
	$\cos 2x(-4q+8p+5q)$ $+\sin 2x(-4p-8q+5p)\ =65\sin 2x$	A1	For correct equation
	$\left.\begin{array}{l}8p+q=0\\ p-8q=65\end{array}\right\}\quad p=1,\quad q=-8$	M1	For equating coefficients of $\cos 2x$ and $\sin 2x$ and attempting to solve for p and/or q
	PI $=\sin 2x-8\cos 2x$	A1	For correct p and q
	$\Rightarrow y=$ $\text{e}^{-2x}(C\cos x+D\sin x)+\sin 2x-8\cos 2x$	B1√ 9	For using GS = CF + PI, with 2 arbitrary constants in CF and none in PI
		9	
5 (i)	$y=u-\dfrac{1}{x}\Rightarrow\dfrac{dy}{dx}=\dfrac{du}{dx}+\dfrac{1}{x^2}$	M1 A1	For differentiating substitution For correct expression
	$x^3\left(\dfrac{du}{dx}+\dfrac{1}{x^2}\right)=x\left(u-\dfrac{1}{x}\right)+x+1$	M1	For substituting y and $\dfrac{dy}{dx}$ into DE
	$\Rightarrow x^2\dfrac{du}{dx}=u$	A1 4	For obtaining correct equation **AG**
(ii)	METHOD 1 $\displaystyle\int\frac{1}{u}du=\int\frac{1}{x^2}dx \Rightarrow \ln ku=-\frac{1}{x}$	M1 A1	For separating variables and attempt at integration For correct integration (k not required here)
	$ku=\text{e}^{-1/x}\Rightarrow k\left(y+\dfrac{1}{x}\right)=\text{e}^{-1/x}$	M1 M1	For any 2 of / For all 3 of } k seen, exponentiating, substituting for u
	$\Rightarrow\ y=A\text{e}^{-1/x}-\dfrac{1}{x}$	A1 5	For correct solution **AEF** in form $y=\text{f}(x)$
	METHOD 2 $\dfrac{du}{dx}-\dfrac{1}{x^2}u=0\Rightarrow$ I.F. $\text{e}^{\int -1/x^2\,dx}=\text{e}^{1/x}$	M1	For attempt to find I.F.
	$\Rightarrow\dfrac{d}{dx}\left(u\,\text{e}^{1/x}\right)=0$	A1	For correct result
	$u\,\text{e}^{1/x}=k\Rightarrow y+\dfrac{1}{x}=k\,\text{e}^{-1/x}$	M1 M1	From $\boxed{u\times\text{I.F.}=}$, for k seen, for substituting for u } in either order
	$\Rightarrow\ y=k\,\text{e}^{-1/x}-\dfrac{1}{x}$	A1	For correct solution **AEF** in form $y=\text{f}(x)$
		9	

6 (i)	METHOD 1 Use 2 of $[-4, 2, 0], [0, 0, 3], [-4, 2, 3], [4, -2, 3]$ or multiples	M1		For finding vector product of 2 appropriate vectors in plane *ACGE*
	$\mathbf{n} = k\,[1, 2, 0]$	A1		For correct **n**
	Use $A[4, 0, 0], C[0, 2, 0], G[0, 2, 3]\ OR\ E[4, 0, 3]$	M1		For substituting a point in the plane
	$\mathbf{r}.[1, 2, 0] = 4$	A1	4	For correct equation. **AEF** in this form
	METHOD 2 $\mathbf{r} = [4, 0, 0] + \lambda[-4, 2, 0] + \mu[0, 0, 3]$	M1		For writing plane in 2-parameter form
	$\Rightarrow x = 4 - 4\lambda,\ y = 2\lambda,\ z = 3\mu$	A1		For 3 correct equations
	$x + 2y = 4$	M1		For eliminating λ (and μ)
	$\Rightarrow \mathbf{r}.[1, 2, 0] = 4$	A1		For correct equation. **AEF** in this form
(ii)	$\theta = \cos^{-1}\dfrac{\lvert[3, 0, -4].[1, 2, 0]\rvert}{\sqrt{3^2+0^2+4^2}\sqrt{1^2+2^2+0^2}}$	B1√ M1 M1		For using correct vectors (allow multiples). f.t. from **n** For using scalar product For multiplying both moduli in denominator
	$\theta = \cos^{-1}\dfrac{3}{5\sqrt{5}} = 74.4°$ $(74.435...°, 1.299...)$	A1	4	For correct angle
(iii)	*AM*: $(\mathbf{r} =)\,[4, 0, 0] + t[-2, 2, 3]$ $(or\ [2, 2, 3] + t[-2, 2, 3])$	M1 A1		For obtaining parametric expression for *AM* For correct expression seen or implied
	$3(4 - 2t) - 4(3t) = 0$ $(or\ 3(2 - 2t) - 4(3 + 3t) = 0)$	M1		For finding intersection of *AM* with *ACGE*
	$t = \frac{2}{3}\ (or\ t = -\frac{1}{3})\ OR\ \mathbf{w} = \left[\frac{8}{3}, \frac{4}{3}, 2\right]$	A1		For correct t *OR* position vector
	$AW : WM = 2 : 1$	A1	5	For correct ratio
		13		
7 (i) (a)	$x + y - a \in \mathbb{R}$	B1		For stating closure is satisfied
	$(x * y) * z = (x + y - a) * z = x + y + z - 2a$	M1		For using 3 distinct elements bracketed both ways
	$x * (y * z) = x * (y + z - a) = x + y + z - 2a$	A1		For obtaining the same result twice for associativity **SR** 3 distinct elements bracketed once, expanded, and symmetry noted scores M1 A1
	$x + e - a = x \Rightarrow e = a$	B1		For stating identity $= a$
	$x + x^{-1} - a = a \Rightarrow x^{-1} = 2a - x$	M1 A1	6	For attempting to obtain inverse of x For obtaining inverse $= 2a - x$ *OR* for showing that inverses exist, where $x + x^{-1} = 2a$
(b)	$x + y - a = y + x - a \Rightarrow$ commutative	B1	1	For stating commutativity is satisfied, with justification
(c)	x order 2 $\Rightarrow x * x = e \Rightarrow 2x - a = e$	M1		For obtaining equation for an element of order 2
	$\Rightarrow 2x - a = a \Rightarrow x = a = e$ *OR* $x = x^{-1} \Rightarrow x = 2a - x \Rightarrow x = a = e$ $\Rightarrow$ no elements of order 2	A1	2	For solving and showing that the only solution is the identity (which has order 1) *OR* For proving that there are no self-inverse elements (other than the identity)

(ii)	e.g. $2+1-5=-2 \notin \mathbb{R}^+$	M1	For attempting to disprove closure
	$\Rightarrow$ not closed	A1	For stating closure is not necessarily satisfied ($0 < x+y \leqslant 5$ required)
	e.g. $2\times 5-11=-1 \notin \mathbb{R}^+$	M1	For attempting to find an element with no inverse
	$\Rightarrow$ no inverse	A1 **4**	For stating inverse is not necessarily satisfied ($x \geqslant 10$ required)
		13	
8 (i)	$\sin\theta = \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta}\right)$	B1	z may be used for $\mathrm{e}^{\mathrm{i}\theta}$ throughout For expression for $\sin\theta$ seen or implied
		M1	For expanding $\left(\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta}\right)^6$
	$\sin^6\theta =$		At least 4 terms and 3 binomial coefficients required.
	$-\frac{1}{64}\left(\mathrm{e}^{6\mathrm{i}\theta}-6\mathrm{e}^{4\mathrm{i}\theta}+15\mathrm{e}^{2\mathrm{i}\theta}-20+15\mathrm{e}^{-2\mathrm{i}\theta}-6\mathrm{e}^{-4\mathrm{i}\theta}+\mathrm{e}^{-6\mathrm{i}\theta}\right)$	A1	For correct expansion. Allow $\frac{\pm(\mathrm{i})}{64}(\ldots\ldots)$
	$=-\frac{1}{64}\left(2\cos 6\theta-12\cos 4\theta+30\cos 2\theta-20\right)$	M1	For grouping terms and using multiple angles
	$\sin^6\theta=-\frac{1}{32}\left(\cos 6\theta-6\cos 4\theta+15\cos 2\theta-10\right)$	A1 **5**	For answer obtained correctly **AG**
(ii)	$\cos^6\theta =$ *OR* $\sin^6\left(\frac{1}{2}\pi-\theta\right)=$	M1	For substituting $\left(\frac{1}{2}\pi-\theta\right)$ for θ throughout
	$-\frac{1}{32}\left(\cos(3\pi-6\theta)-6\cos(2\pi-4\theta)+15\cos(\pi-2\theta)-10\right)$	A1	For correct unsimplified expression
	$\cos^6\theta=\frac{1}{32}\left(\cos 6\theta+6\cos 4\theta+15\cos 2\theta+10\right)$	A1 **3**	For correct expression with $\cos n\theta$ terms **AEF**
(iii)	$\int_0^{\frac{1}{4}\pi}\frac{1}{32}\left(-2\cos 6\theta-30\cos 2\theta\right)\,\mathrm{d}\theta$	B1√	For correct integral. f.t. from $\sin^6\theta-\cos^6\theta$
	$=-\frac{1}{16}\left[\frac{1}{6}\sin 6\theta+\frac{15}{2}\sin 2\theta\right]_0^{\frac{1}{4}\pi}$	M1 A1√	For integrating $\cos n\theta, \sin n\theta$ *or* $\mathrm{e}^{\mathrm{i}n\theta}$ For correct integration. f.t. from integrand
	$=-\frac{11}{24}$	A1 **4**	For correct answer **www**
		12	