OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE UNIT **4726/01**

MATHEMATICS

Further Pure Mathematics 2

THURSDAY 7 JUNE 2007 Morning

Time: 1 hour 30 minutes

Additional Materials: Answer Booklet (8 pages)
List of Formulae (MF1)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.

ADVICE TO CANDIDATES

- Read each question carefully and make sure you know what you have to do before starting your answer.
- **You are reminded of the need for clear presentation in your answers.**

This document consists of **4** printed pages.

 [Turn over

1 The equation of a curve, in polar coordinates, is

$$r = 2\sin 3\theta, \quad \text{for } 0 \leqslant \theta \leqslant \tfrac{1}{3}\pi.$$

Find the exact area of the region enclosed by the curve between $\theta = 0$ and $\theta = \frac{1}{3}\pi$. [4]

2 (i) Given that $f(x) = \sin(2x + \frac{1}{4}\pi)$, show that $f(x) = \frac{1}{2}\sqrt{2}(\sin 2x + \cos 2x)$. [2]

(ii) Hence find the first four terms of the Maclaurin series for $f(x)$. [You may use appropriate results given in the List of Formulae.] [3]

3 It is given that $f(x) = \dfrac{x^2 + 9x}{(x-1)(x^2+9)}$.

(i) Express $f(x)$ in partial fractions. [4]

(ii) Hence find $\int f(x)\, dx$. [2]

4 (i) Given that

$$y = x\sqrt{1 - x^2} - \cos^{-1} x,$$

find $\dfrac{dy}{dx}$ in a simplified form. [4]

(ii) Hence, or otherwise, find the exact value of $\int_0^1 2\sqrt{1 - x^2}\, dx$. [3]

5 It is given that, for non-negative integers n,

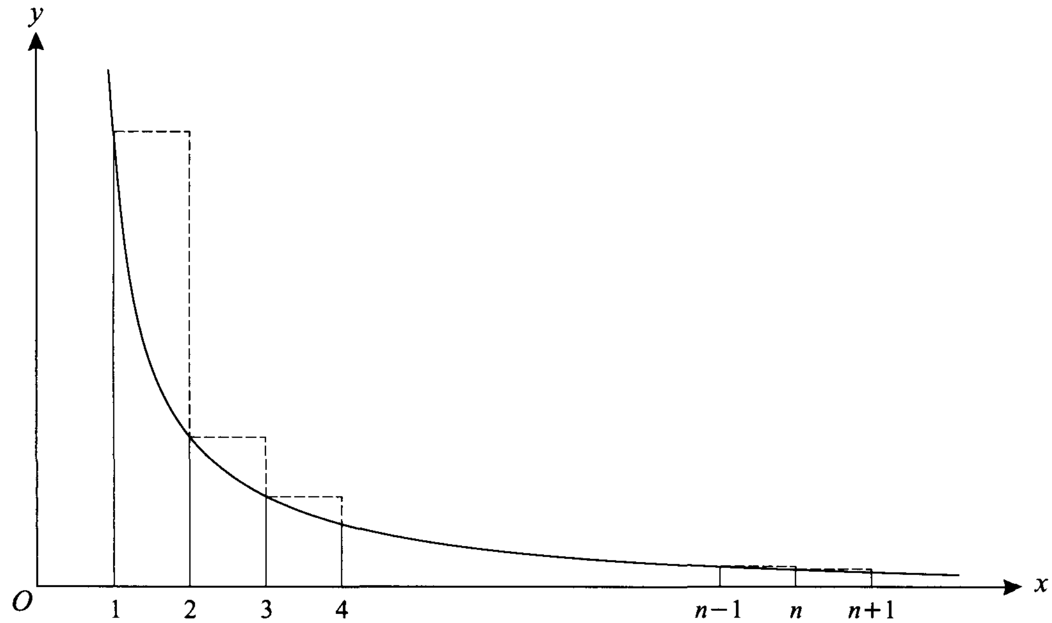
$$I_n = \int_1^e (\ln x)^n\, dx.$$

(i) Show that, for $n \geqslant 1$,

$$I_n = e - nI_{n-1}.$$ [4]

(ii) Find I_3 in terms of e. [4]

6



The diagram shows the curve with equation $y = \frac{1}{x^2}$ for $x > 0$, together with a set of n rectangles of unit width, starting at $x = 1$.

(i) By considering the areas of these rectangles, explain why

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} > \int_1^{n+1} \frac{1}{x^2}\,dx.$$ [2]

(ii) By considering the areas of another set of rectangles, explain why

$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{n^2} < \int_1^{n} \frac{1}{x^2}\,dx.$$ [3]

(iii) Hence show that

$$1 - \frac{1}{n+1} < \sum_{r=1}^{n} \frac{1}{r^2} < 2 - \frac{1}{n}.$$ [4]

(iv) Hence give bounds between which $\sum_{r=1}^{\infty} \frac{1}{r^2}$ lies. [2]

7 **(i)** Using the definitions of hyperbolic functions in terms of exponentials, prove that

$$\cosh x \cosh y - \sinh x \sinh y = \cosh(x - y).$$ [4]

(ii) Given that $\cosh x \cosh y = 9$ and $\sinh x \sinh y = 8$, show that $x = y$. [2]

(iii) Hence find the values of x and y which satisfy the equations given in part **(ii)**, giving the answers in logarithmic form. [4]

8 The iteration $x_{n+1} = \dfrac{1}{(x_n + 2)^2}$, with $x_1 = 0.3$, is to be used to find the real root, α, of the equation $x(x+2)^2 = 1$.

(i) Find the value of α, correct to 4 decimal places. You should show the result of each step of the iteration process. [4]

(ii) Given that $f(x) = \dfrac{1}{(x+2)^2}$, show that $f'(\alpha) \neq 0$. [2]

(iii) The difference, δ_r, between successive approximations is given by $\delta_r = x_{r+1} - x_r$. Find δ_3. [1]

(iv) Given that $\delta_{r+1} \approx f'(\alpha)\delta_r$, find an estimate for δ_{10}. [3]

9 It is given that the equation of a curve is

$$y = \frac{x^2 - 2ax}{x - a},$$

where a is a positive constant.

(i) Find the equations of the asymptotes of the curve. [4]

(ii) Show that y takes all real values. [4]

(iii) Sketch the curve $y = \dfrac{x^2 - 2ax}{x - a}$. [3]

1		Correct formula with correct r	M1	Allow $r^2 = 2\sin^2 3\theta$
		Rewrite as $a + b\cos 6\theta$	M1	$a, b \neq 0$
		Integrate their expression correctly	A1√	From $a + b\cos 6\theta$
		Get $\frac{1}{3}\pi$	A1	cao
2	(i)	Expand to $\sin 2x \cos\frac{1}{4}\pi + \cos 2x \sin\frac{1}{4}\pi$	B1	
		Clearly replace $\cos\frac{1}{4}\pi$, $\sin\frac{1}{4}\pi$ to A.G.	B1	
	(ii)	Attempt to expand $\cos 2x$	M1	Allow $1 - 2x^2/2$
		Attempt to expand $\sin 2x$	M1	Allow $2x - 2x^3/3$
		Get $\frac{1}{2}\sqrt{2}\,(1 + 2x - 2x^2 - 4x^3/3)$	A1	Four correct unsimplified terms in any order; allow bracket; AEEF SR Reasonable attempt at $f^n(0)$ for $n = 0$ to 3 M1 Attempt to replace their values in Maclaurin M1 Get correct answer only A1
3	(i)	Express as $A/(x-1) + (Bx+C)/(x^2+9)$	M1	Allow C=0 here
		Equate (x^2+9x) to $A(x^2+9) + (Bx+C)(x-1)$	M1√	May imply above line; on their P.F.
		Sub. for x or equate coeff.	M1	Must lead to at least 3 coeff.; allow cover-up method for A
		Get A=1, B=0,C=9	A1	cao from correct method
	(ii)	Get $A\ln(x-1)$	B1√	On their A
		Get $C/3 \tan^{-1}(x/3)$	B1√	On their C; condone no constant; ignore any $B \neq 0$
4	(i)	Reasonable attempt at product rule	M1	Two terms seen
		Derive or quote diff. of $\cos^{-1} x$	M1	Allow +
		Get $-x^2(1 - x^2)^{-\frac{1}{2}} + (1 - x^2)^{\frac{1}{2}} + (1 - x^2)^{-\frac{1}{2}}$	A1	
		Tidy to $2(1 - x^2)^{\frac{1}{2}}$	A1	cao
	(ii)	Write down integral from (i)	B1	On any $k\sqrt{(1-x^2)}$
		Use limits correctly	M1	In any reasonable integral
		Tidy to $\frac{1}{2}\pi$	A1	
			SR	Reasonable sub. B1 Replace for new variable and attempt to integrate (ignore limits) M1 Clearly get $\frac{1}{2}\pi$ A1

5	(i)	Attempt at parts on $\int 1\ (\ln x)^n\ dx$	M1	Two terms seen
		Get $x\ (\ln x)^n - \int n\ (\ln x)^{n-1}\ dx$	A1	
		Put in limits correctly in line above	M1	
		Clearly get A.G.	A1	ln e =1, ln1 =0 seen or implied
	(ii)	Attempt I_3 to I_2 as $I_3 = e - 3I_2$	M1	
		Continue sequence in terms of In	A1	$I_2 = e-2I_1$ and/or $I_1 = e-I_0$
		Attempt I_0 or I_1	M1	(I_0=e-1, I_1=1)
		Get 6 – 2e	A1	cao
6	(i)	Area under graph (= $\int 1/x^2\ dx$, 1 to n+1)		
		< Sum of rectangles (from 1 to n)	B1	Sum (total) seen or implied eg diagram; accept areas (of rectangles)
		Area of each rectangle = Width x Height = 1 x $1/x^2$	B1	Some evidence of area worked out – seen or implied
	(ii)	Indication of new set of rectangles	B1	
		Similarly, area under graph from 1 to n > sum of areas of rectangles from 2 to n	B1	Sum (total) seen or implied
		Clear explanation of A.G.	B1	Diagram; use of left-shift of previous areas
	(iii)	Show complete integrations of RHS, using correct, different limits	M1	Reasonable attempt at $\int x^{-2}\ dx$
		Correct answer, using limits, to one integral	A1	
		Add 1 to their second integral to get complete series	M1	
		Clearly arrive at A.G.	A1	
	(iv)	Get one limit	B1	Quotable
		Get both 1 and 2	B1	Quotable; limits only required

7	(i)	Use correct definition of cosh or sinh x	B1	Seen anywhere in (i)
		Attempt to mult. their cosh/sinh	M1	
		Correctly mult. out and tidy	A1√	
		Clearly arrive at A.G.	A1	Accept e^{x-y} and e^{y-x}
	(ii)	Get $\cosh(x-y) = 1$	M1	
		Get or imply $(x-y) = 0$ to A.G.	A1	
	(iii)	Use $\cosh^2 x = 9$ or $\sinh^2 x = 8$	B1	
		Attempt to solve cosh x = 3 (not –3) or sinh $x = \pm\sqrt{8}$ (allow $+\sqrt{8}$ or $-\sqrt{8}$ only)	M1	x = ln(3 + √8) from formulae book or from basic cosh definition
		Get at least one x solution correct	A1	
		Get both solutions correct, x and y	A1	x, y = ln(3 ±2√2); AEEF SR Attempt tanh = sinh/cosh B1 Get tanh x = ±√8/3 (+ or -) M1 Get at least one sol. correct A1 Get both solutions correct A1 SR Use exponential definition B1 Get quadratic in e^x or e^{2x} M1 Solve for one correct x A1 Get both solutions, x and y A1
8	(i)	$x_2 = 0.1890$	B1	
		$x_3 = 0.2087$	B1√	From their x_1 (or any other correct)
		$x_4 = 0.2050$	B1√	Get at least two others correct, all to a minimum of 4 d.p.
		$x_5 = 0.2057$		
		$x_6 = 0.2055$		
		$x_7 (= x_8) = 0.2056$ (to x_7 minimum)		
		$\alpha = 0.2056$	B1	cao; answer may be retrieved despite some errors
	(ii)	Attempt to diff. f(x)	M1	$k/(2+x)^3$
		Use α to show $f'(\alpha) \neq 0$	A1√	Clearly seen, or explain $k/(2+x)^3 \neq 0$ as k ≠ 0; allow ± 0.1864
			SR	Translate $y=1/x^2$ M1 State/show $y=1/x^2$ has no TP A1
	(iii)	$\delta_3 = -0.0037$ (allow –0.004)	B1√	Allow ±, from their x_4 and x_3
	(iv)	Develop from $\delta_{10} = f'(\alpha)\,\delta_9$ etc. to get δ_i or quote $\delta_{10} = \delta_3\, f'(\alpha)^7$	M1	Or any δ_i eg use $\delta_9 = x_{10} - x_9$
		Use their δ_i and $f'(\alpha)$	M1	
		Get 0.000000028	A1	Or answer that rounds to ± 0.00000003

9	(i)	Quote $x = a$	B1	
		Attempt to divide out	M1	Allow M1 for y=x here; allow
			A1	$(x-a) + k/(x-a)$ seen or implied
		Get $y = x - a$	A1	Must be equations
	(ii)	Attempt at quad. in x (=0)	M1	
		Use $b^2 - 4ac \geq 0$ for real x	M1	Allow >
		Get $y^2 + 4a^2 \geq 0$	A1	
		State/show their quad. is always >0	B1	Allow ≥
	(iii)		B1√	Two asymptotes from (i) (need not be labelled)
			B1	Both crossing points
			B1√	Approaches – correct shape
			SR	Attempt diff. by quotient/product rule M1
				Get quadratic in x for $dy/dx = 0$ and note $b^2 - 4ac < 0$ A1
				Consider horizontal asymptotes B1
				Fully justify answer B1