OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MATHEMATICS **4726**

Further Pure Mathematics 2

Tuesday **6 JUNE 2006** Afternoon 1 hour 30 minutes

Additional materials:
8 page answer booklet
Graph paper
List of Formulae (MF1)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.
- Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.
- **You are reminded of the need for clear presentation in your answers.**

This question paper consists of 4 printed pages.

1 Find the first three non-zero terms of the Maclaurin series for

$$(1+x)\sin x,$$

simplifying the coefficients. [3]

2 (i) Given that $y = \tan^{-1} x$, prove that $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$. [3]

(ii) Verify that $y = \tan^{-1} x$ satisfies the equation

$$(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0.$$ [3]

3 The equation of a curve is $y = \dfrac{x+1}{x^2+3}$.

(i) State the equation of the asymptote of the curve. [1]

(ii) Show that $-\frac{1}{6} \leqslant y \leqslant \frac{1}{2}$. [5]

4 (i) Using the definition of $\cosh x$ in terms of e^x and e^{-x}, prove that

$$\cosh 2x = 2\cosh^2 x - 1.$$ [3]

(ii) Hence solve the equation

$$\cosh 2x - 7\cosh x = 3,$$

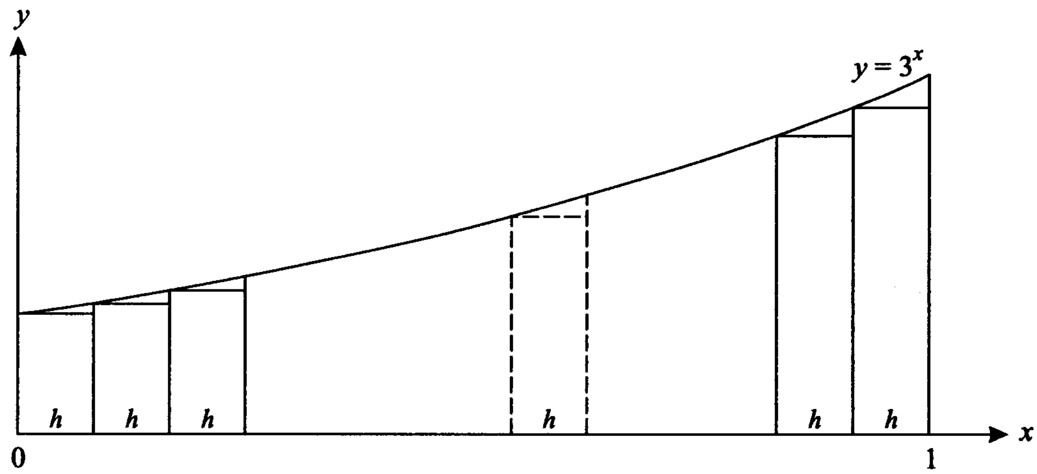
giving your answer in logarithmic form. [4]

5 (i) Express $t^2 + t + 1$ in the form $(t+a)^2 + b$. [1]

(ii) By using the substitution $\tan \frac{1}{2}x = t$, show that

$$\int_0^{\frac{1}{2}\pi} \frac{1}{2+\sin x}\,dx = \frac{\sqrt{3}}{9}\pi.$$ [6]

6



The diagram shows the curve with equation $y = 3^x$ for $0 \leqslant x \leqslant 1$. The area A under the curve between these limits is divided into n strips, each of width h where $nh = 1$.

(i) By using the set of rectangles indicated on the diagram, show that $A > \dfrac{2h}{3^h - 1}$. [3]

(ii) By considering another set of rectangles, show that $A < \dfrac{(2h)3^h}{3^h - 1}$. [3]

(iii) Given that $h = 0.001$, use these inequalities to find values between which A lies. [2]

7 The equation of a curve, in polar coordinates, is

$$r = \sqrt{3} + \tan\theta, \quad \text{for } -\tfrac{1}{3}\pi \leqslant \theta \leqslant \tfrac{1}{4}\pi.$$

(i) Find the equation of the tangent at the pole. [2]

(ii) State the greatest value of r and the corresponding value of θ. [2]

(iii) Sketch the curve. [2]

(iv) Find the exact area of the region enclosed by the curve and the lines $\theta = 0$ and $\theta = \frac{1}{4}\pi$. [5]

8 The curve with equation $y = \dfrac{\sinh x}{x^2}$, for $x > 0$, has one turning point.

(i) Show that the x-coordinate of the turning point satisfies the equation $x - 2\tanh x = 0$. [3]

(ii) Use the Newton-Raphson method, with a first approximation $x_1 = 2$, to find the next two approximations, x_2 and x_3, to the positive root of $x - 2\tanh x = 0$. [5]

(iii) By considering the approximate errors in x_1 and x_2, estimate the error in x_3. (You are not expected to evaluate x_4.) [3]

[Question 9 is printed overleaf.]

[Turn over

9 **(i)** Given that $y = \sinh^{-1} x$, prove that $y = \ln\left(x + \sqrt{x^2 + 1}\right)$. [3]

(ii) It is given that, for non-negative integers n,

$$I_n = \int_0^{\alpha} \sinh^n \theta \, d\theta,$$

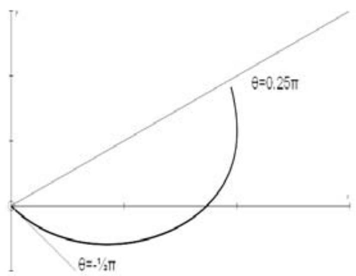
where $\alpha = \sinh^{-1} 1$. Show that

$$nI_n = \sqrt{2} - (n - 1)I_{n-2}, \quad \text{for } n \geqslant 2.$$ [6]

(iii) Evaluate I_4, giving your answer in terms of $\sqrt{2}$ and logarithms. [4]

1 Correct expansion of sin x	B1	Quote or derive $x-{}^1/_6x^3$
Multiply their expansion by $(1 + x)$	M1	Ignore extra terms
Obtain $x + x^2 - x^3/6$	A1√	On their sin x; ignore extra terms; allow 3!
	SC	Attempt product rule M1 Attempt f(0), f′(0), f″(0) … (at least 3) M1 Use Maclaurin accurately cao A1
2 (i) Get $\sec^2 y \frac{dy}{dx} = 1$ or equivalent	M1	
Clearly use $1 + \tan^2 y = \sec^2 y$	M1	May be implied
Clearly arrive at A.G.	A1	
(ii) Reasonable attempt to diff. to $\frac{-2x}{(1+x^2)^2}$	M1	Use of chain/quotient rule
Substitute their expressions into D.E.	M1	Or attempt to derive diff. equ^n.
Clearly arrive at A.G.	A1	
	SC	Attempt diff. of $(1+x^2)\frac{dy}{dx} = 1$ M1,A1 Clearly arrive at A.G. B1
3 (i) State $y = 0$ (or seen if working given)	B1	Must be = ; accept x-axis; ignore any others
(ii) Write as quad. in x^2	M1	$(x^2y - x + (3y-1) = 0)$
Use for real x, $b^2-4ac \geq 0$	M1	Allow > ; or < for no real x
Produce quad. inequality in y	M1	$1 \geq 12y^2 - 4y$; $12y^2 - 4y - 1 \leq 0$
Attempt to solve inequality	M1	Factorise/ quadratic formula
Justify A.G.	A1	e.g. diagram / table of values of y
	SC	Attempt diff. by product/quotient M1 Solve $dy/dx = 0$ for two real x M1 Get both $(-3,-{}^1/_6)$ and $(1,½)$ A1 Clearly prove min./max. A1 Justify fully the inequality e.g. detailed graph B1
4 (i) Correct definition of cosh x or cosh $2x$	B1	
Attempt to sub. in RHS and simplify	M1	or LHS if used
Clearly produce A.G.	A1	
(ii) Write as quadratic in cosh x	M1	$(2\cosh^2 x - 7\cosh x - 4 = 0)$
Solve their quadratic accurately	A1√	Factorise/quadratic formula
Justify one answer only	B1	State $\cosh x \geq 1$/graph; allow ≥ 0
Give $\ln(4 + \sqrt{15})$	A1	cao; any one of $\pm \ln(4 \pm \sqrt{15})$ or decimal equivalent of ln ( )
5 (i) Get $(t + ½)^2 + ¾$	B1	cao
(ii) Derive or quote $dx = \frac{2}{1+t^2} dt$	B1	
Derive or quote $\sin x = 2t/(1 + t^2)$	B1	
Attempt to replace all x and dx	M1	
Get integral of form $A/(Bt^2+Ct+D)$	A1√	From their expressions, $C \neq 0$
Use complete square form as $\tan^{-1}(f(t))$	M1	From formulae book or substitution
Get A.G.	A1	

6 (i) Attempt to sum areas of rectangles	M1	$(h.3^h + h.3^{2h} + \ldots + h.3^{(n-1)h})$
Use G.P. on $h(1+3^h+3^{2h}+\ldots+3^{(n-1)h})$	M1	All terms not required, but last term needed (or 3^{1-h}); or specify a, r and n for a G.P.
Simplify to A.G.	A1	Clearly use $nh = 1$
(ii) Attempt to find sum areas of different rect.	M1	Different from (i)
Use G.P. on $h(3^h+3^{2h}+\ldots+3^{nh})$	M1	All terms not required, but last term needed; G.P. specified as in (i), or deduced from (i)
Simplify to A.G.	A1	
(iii) Get 1.8194(8), 1.8214(8) correct	B1,B1	Allow $1.81 \le A \le 1.83$
7 (i) Attempt to solve r=0, $\tan\theta = -\sqrt{3}$	M1	Allow $\pm\sqrt{3}$
Get $\theta = -\frac{1}{3}\pi$ only	A1	Allow -60°
(ii) $r = \sqrt{3} + 1$ when $\theta = \frac{1}{4}\pi$	B1,B1	AEF for r, 45° for θ
(iii)	B1	Correct r at correct end-values of θ; Ignore extra θ used
	B1	Correct shape with r not decreasing
(iv) Formula with correct r used	M1	r^2 may be implied
Replace $\tan^2\theta = \sec^2\theta - 1$	B1	
Attempt to integrate <u>their</u> expression	M1	Must be 3 different terms leading to any 2 of $a\theta + b\ln(\sec\theta/\cos\theta) + c\tan\theta$
Get $\theta + \sqrt{3}\ln\sec\theta + \frac{1}{2}\tan\theta$	A1	Condone answer x2 if ½ seen elsewhere
Correct limits to $\frac{1}{4}\pi + \sqrt{3}\ln\sqrt{2} + \frac{1}{2}$	A1	cao; AEF
8 (i) Attempt to diff. using product/quotient	M1	
Attempt to solve $dy/dx = 0$	M1	
Rewrite as A.G.	A1	Clearly gain A.G.
(ii) Diff. to $f'(x) = 1 \pm 2\,\text{sech}^2 x$	B1	Or $\pm 2\,\text{sech}^2 x - 1$
Use correct form of N-R with their expressions from correct f(x)	M1	
Attempt N-R with x_1= 2 from previous M1	M1	To get an x_2
Get x_2 = 1.9162(2) (3 s.f. min.)	A1	
Get x_3 = 1.9150(1) (3 s.f. min.)	A1	cao
(iii) Work out e_1 and e_2 (may be implied)	B1√	-0.083(8), -0.0012 (allow ± if both of same sign); e_1 from 0.083 to 0.085
Use $e_2 \approx k e_1^2$ and $e_3 \approx k e_2^2$	M1	
Get $e_3 \approx e_2^3/e_1^2 = -0.0000002$(or 3)	A1√	± if same sign as B1√
	SC	B1 only for $x_4 - x_3$



9 (i) Rewrite as quad. in e^y	M1 Any form
Solve to $e^y = (x \pm \sqrt{(x^2+1)})$	A1 Allow $y = \ln(\quad)$
Justify one solution only	B1 $x - \sqrt{(x^2+1)} < 0$ for all real x
	SC Use $C^2 - S^2 = 1$ for $C = \pm\sqrt{(1+x^2)}$ M1
	Use/state $\cosh y + \sinh y = e^y$ A1
	Justify one solution only B1
(ii) Attempt parts on $\sinh x . \sinh^{n-1} x$	M1
Get correct answer	A1 $(\cosh x . \sinh^{n-1} x - \int \cosh^2 x . (n-1)\sinh^{n-2} x \, dx)$
Justify $\sqrt{2}$ by $\sqrt{(1+\sinh^2 x)}$ for $\cosh x$ when limits inserted	B1 Must be clear
Replace $\cosh^2 = 1 + \sinh^2$; tidy at this stage	M1
Produce I_{n-2}	A1
Gain A.G. clearly	A1
(iii) Attempt $4I_4 = \sqrt{2} - 3I_2$,$2I_2 = \sqrt{2} - I_0$	M1 Clear attempt at iteration (one at least seen)
Work out $I_0 = \sinh^{-1} 1 = \ln(1+\sqrt{2}) = \alpha$	B1 Allow I_2
Sub. back completely for I_4	M1
Get $^1/_8(3\ln(1+\sqrt{2}) - \sqrt{2})$	A1 AEEF