OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS (MEI) 4769

Statistics 4

Candidates answer on the answer booklet.

OCR supplied materials:
- 8 page answer booklet
  (sent with general stationery)
- MEI Examination Formulae and Tables (MF2)

Other materials required:
- Scientific or graphical calculator

Thursday 26 May 2011
Morning

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet. Please write clearly and in capital letters.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer any **three** questions.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

Option 1: Estimation

1 The random variable X has the Normal distribution with mean 0 and variance θ, so that its probability density function is

$$f(x) = \frac{1}{\sqrt{2\pi\theta}} e^{-x^2/2\theta}, \quad -\infty < x < \infty,$$

where θ ($\theta > 0$) is unknown. A random sample of n observations from X is denoted by $X_1, X_2, \ldots, X_n$.

(i) Find $\hat{\theta}$, the maximum likelihood estimator of θ. **[14]**

(ii) Show that $\hat{\theta}$ is an unbiased estimator of θ. **[4]**

(iii) In large samples, the variance of $\hat{\theta}$ may be estimated by $\frac{2\hat{\theta}^2}{n}$. Use this and the results of parts **(i)** and **(ii)** to find an approximate 95% confidence interval for θ in the case when $n = 100$ and $\Sigma X_i^2 = 1000$. **[6]**

Option 2: Generating Functions

2 The random variable X has the χ_n^2 distribution. This distribution has moment generating function $M(\theta) = (1 - 2\theta)^{-\frac{1}{2}n}$, where $\theta < \frac{1}{2}$.

(i) Verify the expression for $M(\theta)$ quoted above for the cases $n = 2$ and $n = 4$, given that the probability density functions of X in these cases are as follows. **[10]**

$n = 2$: $\quad f(x) = \frac{1}{2}e^{-\frac{1}{2}x} \quad (x > 0)$

$n = 4$: $\quad f(x) = \frac{1}{4}xe^{-\frac{1}{2}x} \quad (x > 0)$

(ii) For the general case, use $M(\theta)$ to find the mean and variance of X in terms of n. **[7]**

(iii) $Y_1, Y_2, \ldots, Y_k$ are independent random variables, each with the χ_1^2 distribution. Show that $W = \sum_{i=1}^{k} Y_i$ has the χ_k^2 distribution. **[4]**

(iv) Use the Central Limit Theorem to find an approximation for $P(W < 118.5)$ for the case $k = 100$. **[3]**

Option 3: Inference

3 **(i)** Explain the meaning of the following terms in the context of hypothesis testing: Type I error, Type II error, operating characteristic, power. **[8]**

(ii) A market research organisation is designing a sample survey to investigate whether expenditure on everyday food items has increased in 2011 compared with 2010. For one of the populations being studied, the random variable X is used to model weekly expenditure, in £, on these items in 2011, where X is Normally distributed with mean μ and variance σ^2. As the corresponding mean value in 2010 was 94, the hypotheses to be examined are

H_0: $\mu = 94$,

H_1: $\mu > 94$.

By comparison with the corresponding 2010 value, σ^2 is assumed to be 25.

The following criteria for the survey are laid down.

- If in fact $\mu = 94$, the probability of concluding that $\mu > 94$ must be only 2%
- If in fact $\mu = 97$, the probability of concluding that $\mu > 94$ must be 95%

A random sample of size n is to be taken and the usual Normal test based on $\overline{X}$ is to be used, with a critical value of c such that H_0 is rejected if the value of $\overline{X}$ exceeds c. Find c and the smallest value of n that is required. **[13]**

(iii) Sketch the power function of an ideal test for examining the hypotheses in part **(ii)**. **[3]**

Option 4: Design and Analysis of Experiments

4 **(a)** Provide an example of an experimental situation where there is one factor of primary interest and where a suitable experimental design would be

(i) randomised blocks,

(ii) a Latin square.

In each case, explain carefully why the design is suitable and why the other design would not be appropriate. **[12]**

(b) An industrial experiment to compare four treatments for increasing the tensile strength of steel is carried out according to a completely randomised design. For various reasons, it is not possible to use the same number of replicates for each treatment. The increases, in a suitable unit of tensile strength, are as follows.

Treatment A	Treatment B	Treatment C	Treatment D
10.1	21.1	9.2	22.6
21.2	20.3	8.8	17.4
11.6	16.0	15.2	23.1
13.6		15.0	19.2
		12.4	

[The sum of these data items is 256.8 and the sum of their squares is 4471.92.]

Construct the usual one-way analysis of variance table. Carry out the appropriate test, using a 5% significance level. **[12]**

THERE ARE NO QUESTIONS PRINTED ON THIS PAGE

RECOGNISING ACHIEVEMENT

GCE

Mathematics (MEI)

Advanced GCE

Unit **4769:** Statistics 4

Mark Scheme for June 2011

This mark scheme is published as an aid to teachers and students, to indicate the requirements of the examination. It shows the basis on which marks were awarded by Examiners. It does not indicate the details of the discussions which took place at an Examiners' meeting before marking commenced.

All Examiners are instructed that alternative correct answers and unexpected approaches in candidates' scripts must be given marks that fairly reflect the relevant knowledge and skills demonstrated.

Mark schemes should be read in conjunction with the published question papers and the Report on the Examination.

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4769 June 2011 Qu 1

$\mathrm{f}(x) = \dfrac{1}{\sqrt{2\pi\theta}} \mathrm{e}^{-x^2/2\theta}$ [N(0, θ)]	
(i) $L = \dfrac{1}{\sqrt{2\pi\theta}} \mathrm{e}^{-x_1^2/2\theta} \cdot \dfrac{1}{\sqrt{2\pi\theta}} \mathrm{e}^{-x_2^2/2\theta} \cdot \ldots \cdot \dfrac{1}{\sqrt{2\pi\theta}} \mathrm{e}^{-x_n^2/2\theta}$ $\left[= (2\pi\theta)^{-n/2} \mathrm{e}^{-\Sigma x_i^2/2\theta} \right]$ $\ln L = -\dfrac{n}{2}\ln(2\pi\theta) - \dfrac{1}{2\theta}\sum x_i^2$ $\dfrac{\mathrm{d}\ln L}{\mathrm{d}\theta} = -\dfrac{n}{2}\cdot\dfrac{1}{\theta} + \dfrac{1}{2\theta^2}\sum x_i^2$ $\dfrac{\mathrm{d}\ln L}{\mathrm{d}\theta} = 0$ gives $\dfrac{n}{2\hat{\theta}} = \dfrac{1}{2\hat{\theta}^2}\sum x_i^2$ i.e. $\hat{\theta} = \dfrac{1}{n}\sum x_i^2$ Check this is a maximum. Eg: $\dfrac{\mathrm{d}^2\ln L}{\mathrm{d}\theta^2} = \dfrac{n}{2}\cdot\dfrac{1}{\theta^2} - \dfrac{1}{\theta^3}\sum x_i^2$ which, for $\theta = \hat{\theta}$, is $\dfrac{n}{2\hat{\theta}^2} - \dfrac{n}{\hat{\theta}^2} = -\dfrac{n}{2\hat{\theta}^2} < 0$.	M1 product form A1 fully correct Note. This A1 mark and the next five A1 marks depend on *all* preceding M marks having been earned. M1 for ln *L* A1 fully correct M1 for differentiating A1, A1 for each term M1 A1 A1 M1 A1 A1 for expression involving $\hat{\theta}$ A1 for showing < 0 **[14]**
(ii) First consider E(X^2) = Var(X) + {E(X)}^{2} = θ + 0 $\therefore \mathrm{E}(\hat{\theta}) = \dfrac{1}{n}(\theta + \theta + \ldots + \theta) = \theta$ i.e. $\hat{\theta}$ is unbiased.	M1 A1 A1 A1 **[4]**
(iii) Here $\hat{\theta} = 10$ and Est Var$(\hat{\theta})$ = 2 × 10^2/100 = 2 Approximate confidence interval is given by $10 \pm 1.96\sqrt{2} = 10 \pm 2.77$, i.e. it is (7.23, 12.77).	B1, B1 M1 centred at 10 B1 1.96 M1 Use of √2 A1 c.a.o. Final interval **[6]**

4769 June 2011 Qu 2

(i) $n = 2$ $\quad \mathrm{f}(x) = \frac{1}{2}\mathrm{e}^{-x/2}$	
$\mathrm{M}(\theta) = \mathrm{E}\left(\mathrm{e}^{\theta X}\right) = \int_0^\infty \frac{1}{2}\mathrm{e}^{-x\left(\frac{1}{2}-\theta\right)}\mathrm{d}x$	A1 Any equivalent form
$= \frac{1}{2}\left[\frac{e^{-x\left(\frac{1}{2}-\theta\right)}}{-\left(\frac{1}{2}-\theta\right)}\right]_0^\infty$ **[A1]** $= \frac{\frac{1}{2}}{\frac{1}{2}-\theta}$ **[A1]** $= (1-2\theta)^{-1}$ **[A1]**	A1, A1, A1 for each expression, as shown, **beware printed answer**
$n = 4$ $\quad \mathrm{f}(x) = \frac{1}{4}x\mathrm{e}^{-x/2}$	
$\mathrm{M}(\theta) = \int_0^\infty \frac{1}{4}x\mathrm{e}^{-x\left(\frac{1}{2}-\theta\right)}\mathrm{d}x$	M1 for attempt to integrate this by parts
$= \frac{1}{4}\left\{\left[\frac{x\mathrm{e}^{-x\left(\frac{1}{2}-\theta\right)}}{-\left(\frac{1}{2}-\theta\right)}\right]_0^\infty \textbf{[A1]} - \int_0^\infty \frac{\mathrm{e}^{-x\left(\frac{1}{2}-\theta\right)}}{-\left(\frac{1}{2}-\theta\right)}\mathrm{d}x \textbf{ [A1]}\right\}$	A1, A1 for each component, as shown
$= \frac{1}{4}\left\{[0-0] \textbf{ [A1]} + \frac{1}{\frac{1}{2}-\theta}.2(1-2\theta)^{-1} \textbf{ [A1]}\right\}$	A1, A1 for each component, as shown
$= \frac{1}{2}\frac{1}{\frac{1}{2}(1-2\theta)}(1-2\theta)^{-1} = (1-2\theta)^{-2}$	A1 for final answer, **beware printed answer**
	[10]
(ii) Mean = M'(0) $\quad \mathrm{M}'(\theta) = -2\left(-\frac{n}{2}\right)(1-2\theta)^{-\frac{n}{2}-1} = n(1-2\theta)^{-\frac{n}{2}-1}$	M1 A1
∴ mean = n	A1
Variance = M''(0) – {M'(0)}^{2}	
$\mathrm{M}''(\theta) = n\left(-\frac{n}{2}-1\right)(-2)(1-2\theta)^{-\frac{n}{2}-2} = n(n+2)(1-2\theta)^{-\frac{n}{2}-2}$	M1 A1
∴ M''(0) = $n(n + 2)$	A1
∴ variance = $n(n + 2) - n^2 = 2n$	A1
[**Note.** This part of the question may also be done by expanding the mgf.]	**[7]**

Solution continued on next page

4769 June 2011 Qu 2 **continued**

(iii)	By convolution theorem,	M1
	$$M_W(\theta) = \left\{(1-2\theta)^{-\frac{1}{2}}\right\}^k = (1-2\theta)^{-k/2}.$$	B1
	This is the mgf of χ^2_k,	
	so (by uniqueness of mgfs)	M1
	$W \sim \chi^2_k$.	B1
		[4]
(iv)	$W \sim \chi^2_{100}$ has mean 100, variance 200. Can regard W as the sum of a large "random sample" of χ^2_1 variates.	
	$\therefore \mathrm{P}\left(\chi^2_{100} < 118.5\right) \approx \mathrm{P}\left(\mathrm{N}(0,1) < \dfrac{118.5-100}{\sqrt{200}} = 1.308\right)$	M1 for use of N(0,1) A1 c.a.o. for 1.308
	= 0.9045.	A1 c.a.o.
		[3]

4769 June 2011 Qu 3

(i) Type I error: rejecting null hypothesis **[B1]** when it is true **[B1]** Type II error: accepting null hypothesis **[B1]** when it is false **[B1]** OC: P(accepting null hypothesis **[B1]** as a function of the parameter under investigation **[B1]**) Power: P(rejecting null hypothesis **[B1]** as a function of the parameter under investigation **[B1]**)	8 separate B1 marks for components of answer, as shown Allow B1 out of 2 for P(...) Allow B1 out of 2 for P(...) P(Type II error | the true value of the parameter) scores B1+B1 P(Type I error | the true value of the parameter) scores B1+B1. "1 – OC" as definition scores zero. **[8]**
(ii) $X \sim \mathrm{N}(\mu, 25)$ H_0: $\mu = 94$ H_1: $\mu > 94$ We require $0.02 = \mathrm{P}\left(\text{reject } \mathrm{H}_0 \mid \mu = 94\right) = \mathrm{P}\left(\bar{X} > c \mid \mu = 94\right)$ $= \mathrm{P}\left(\mathrm{N}(94, 25/n) > c\right) = \mathrm{P}\left(\mathrm{N}(0,1) > \dfrac{c-94}{5/\sqrt{n}}\right)$ $\therefore \dfrac{c-94}{5/\sqrt{n}} = 2.054$ We also require $0.95 = \mathrm{P}\left(\text{reject } \mathrm{H}_0 \mid \mu = 97\right)$ $= \mathrm{P}\left(\mathrm{N}(97, 25/n) > c\right) = \mathrm{P}\left(\mathrm{N}(0,1) > \dfrac{c-97}{5/\sqrt{n}}\right)$ $\therefore \dfrac{c-97}{5/\sqrt{n}} = -1.645$ $\therefore$ we have $c = 94 + \dfrac{10.27}{\sqrt{n}}$ and $c = 97 - \dfrac{8.225}{\sqrt{n}}$ Attempt to solve; $c = 95.666$ [allow 95.7 or awrt] $\sqrt{n} = 6.165$, $n = 38.01$ Take n as "next integer up" from candidate's value	 M1 M1 for first expression M1 for standardising B1 for 2.054 M1 for first expression M1 for standardising B1 for –1.645 M1 two equations A1 both correct (FT any previous errors) M1 A1 c.a.o. A1 c.a.o. A1 **[13]**
(iii) Power function: step function from 0 with step marked at 94 to height marked as 1	G1 G1 G1 Zero out of 3 if step is wrong way round. **[3]**

4769 June 2011 Qu 4

(a)	Each E2 in this part is available as E2, E1, E0.	
(i)	Description of situation where randomised blocks would be suitable, ie one extraneous factor (eg stream down one side of a field).	E2
	Explanation of why RB is suitable (the design allows the extraneous factor to be "taken out "separately).	E2
	Explanation of why LS is not appropriate (eg: there is only one extraneous factor; LS would be unnecessarily complicated; not enough degrees of freedom would remain for a sensible estimate of experimental error).	E2
(ii)	Description of situation where Latin square would be suitable, ie two extraneous factors (and all with same number of levels) (eg streams down two sides of a field).	E2
	Explanation of why LS is suitable (the design allows the extraneous factors to be "taken out "separately).	E2
	Explanation of why RB is not appropriate (RB cannot cope with two extraneous factors).	E2 **[12]**

(b) Totals are 56.5 57.4 60.6 82.3 from samples of sizes 4 3 5 4

Grand total 256.8 "Correction factor" CF = $256.8^2/16 = 4121.64$

Total SS = 4471.92 – CF = 350.28

Between treatments SS = $\frac{56.5^2}{4}+\frac{57.4^2}{3}+\frac{60.6^2}{5}+\frac{82.3^2}{4}$ – CF

= 4324.1103 – CF = 202.47

Residual SS (by subtraction) = 350.28 – 202.47 = 147.81

M1 for attempt to form three sums of squares.
M1 for correct method for any two.
A1 if each calculated SS is correct.

Source of variation	SS	df	MS **[M1]**	MS ratio **[M1]**
Between treatments	202.47	3 **[B1]**	67.49	5.47(92) **[A1 cao]**
Residual	147.81	12 **[B1]**	12.3175	
Total	350.28	15		

5 marks within the table, as shown

Refer MS ratio to $F_{3,12}$. — M1 No FT if wrong
Upper 5% point is 3.49. — A1 No FT if wrong
Significant. — E1
Seems the effects of the treatments are not all the same. — E1

[12]

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4769: Statistics 4

General Comments

There were only 18 candidates for this module this year, thinly spread over 9 centres. This is a much smaller entry than last year.

There was much good work with many candidates scoring highly, but comparatively little that was really outstanding.

As usual, the paper consisted of four questions, each within a defined "option" area of the specification. The rubric requires that three be attempted. Two candidates in fact attempted all four. The best three attempts counted. In general, attempting all four questions is not a good strategy; it is better to try to complete three questions. All the questions attracted a reasonable number of attempts, with question 4, on design and analysis of experiments, the least popular though not by very much.

Comments on Individual Questions

1 This was on the "estimation" option. It was based on maximum likelihood estimation.

Part (i), on finding a maximum likelihood estimator, inevitably involved quite a lot of technical work. This was mostly well done, though a few candidates did not know how to form the likelihood to start with (a few follow-through marks were available for subsequent methods). Some candidates had difficulty showing that the obtained turning-point is indeed a maximum; in this case, the actual estimator has to be inserted in the second derivative.

Part (ii) required candidates to show that (in this case – it is *not* true in general) the maximum likelihood estimator is unbiased. Mostly this was done well, but a few candidates became badly lost in confusion between sample and population quantities.

Part (iii) required candidates to obtain an approximate 95% confidence interval for the parameter, using a given result for the variance of the estimator. Again this was mostly done well, but there were some very bad errors of introducing "$\sigma / \sqrt{n}$" in the denominator.

2 This was on the "generating functions" option and was concerned with moment generating functions of chi-squared distributions.

There was good technical work here. It was pleasing to see integrals carefully set out in correct and full notation and with proper attention to insertion of limits, and likewise pleasing to see careful differentiation in part (ii) (not too many cases of a disappearing minus sign). Part (iii) required candidates to invoke fairly explicitly the uniqueness of the relationship between a distribution and its moment generating function. Part (iv) was an application of the Central Limit Theorem; most candidates knew how to use the Normal distribution here, but there were some strange errors with the parameters.

3 This question was on the "inference" option.

It started by requiring definitions of Type I error, Type II error, operating characteristic and power. Sadly there were still candidates who had Type I and Type II errors the wrong way round, which is a bad mistake at this level. In the case of the power, a statement that "power = 1 – operating characteristic" was not accepted as a *definition* of power.

Part (ii) required a critical value and the minimum sample size to be found for a Normal test for the mean, given some criteria for the errors. This was commonly done well.

Part (iii) required a sketch of an ideal power function. Some quite extraordinary sketches came forward here, completely wrong and in some cases simply bizarre, even from candidates who had met with reasonable success in the earlier parts.

4 This was on the "design and analysis of experiments" option.

The first part required candidates to discuss and compare the randomised blocks design and the Latin square design, giving an example for each situation. Mostly this was done fairly well, but candidates were not always completely sound about how these designs can "allow for" one or two extraneous factors.

The second part required an analysis of variance to be carried out, which was usually done efficiently and correctly.

GCE Mathematics (MEI)

			Max Mark	a	b	c	d	e	u
4751/01	(C1) MEI Introduction to Advanced Mathematics	Raw	72	55	49	43	37	32	0
		UMS	100	80	70	60	50	40	0
4752/01	(C2) MEI Concepts for Advanced Mathematics	Raw	72	53	46	39	33	27	0
		UMS	100	80	70	60	50	40	0
4753/01	(C3) MEI Methods for Advanced Mathematics with Coursework: Written Paper	Raw	72	54	48	42	36	29	0
4753/02	(C3) MEI Methods for Advanced Mathematics with Coursework: Coursework	Raw	18	15	13	11	9	8	0
4753/82	(C3) MEI Methods for Advanced Mathematics with Coursework: Carried Forward Coursework Mark	Raw	18	15	13	11	9	8	0
4753	(C3) MEI Methods for Advanced Mathematics with Coursework	UMS	100	80	70	60	50	40	0
4754/01	(C4) MEI Applications of Advanced Mathematics	Raw	90	63	56	50	44	38	0
		UMS	100	80	70	60	50	40	0
4755/01	(FP1) MEI Further Concepts for Advanced Mathematics	Raw	72	59	52	45	39	33	0
		UMS	100	80	70	60	50	40	0
4756/01	(FP2) MEI Further Methods for Advanced Mathematics	Raw	72	55	48	41	34	27	0
		UMS	100	80	70	60	50	40	0
4757/01	(FP3) MEI Further Applications of Advanced Mathematics	Raw	72	55	48	42	36	30	0
		UMS	100	80	70	60	50	40	0
4758/01	(DE) MEI Differential Equations with Coursework: Written Paper	Raw	72	63	57	51	45	39	0
4758/02	(DE) MEI Differential Equations with Coursework: Coursework	Raw	18	15	13	11	9	8	0
4758/82	(DE) MEI Differential Equations with Coursework: Carried Forward Coursework Mark	Raw	18	15	13	11	9	8	0
4758	(DE) MEI Differential Equations with Coursework	UMS	100	80	70	60	50	40	0
4761/01	(M1) MEI Mechanics 1	Raw	72	60	52	44	36	28	0
		UMS	100	80	70	60	50	40	0
4762/01	(M2) MEI Mechanics 2	Raw	72	64	57	51	45	39	0
		UMS	100	80	70	60	50	40	0
4763/01	(M3) MEI Mechanics 3	Raw	72	59	51	43	35	27	0
		UMS	100	80	70	60	50	40	0
4764/01	(M4) MEI Mechanics 4	Raw	72	54	47	40	33	26	0
		UMS	100	80	70	60	50	40	0
4766/01	(S1) MEI Statistics 1	Raw	72	53	45	38	31	24	0
		UMS	100	80	70	60	50	40	0
4767/01	(S2) MEI Statistics 2	Raw	72	60	53	46	39	33	0
		UMS	100	80	70	60	50	40	0
4768/01	(S3) MEI Statistics 3	Raw	72	56	49	42	35	28	0
		UMS	100	80	70	60	50	40	0
4769/01	(S4) MEI Statistics 4	Raw	72	56	49	42	35	28	0
		UMS	100	80	70	60	50	40	0
4771/01	(D1) MEI Decision Mathematics 1	Raw	72	51	45	39	33	27	0
		UMS	100	80	70	60	50	40	0
4772/01	(D2) MEI Decision Mathematics 2	Raw	72	58	53	48	43	39	0
		UMS	100	80	70	60	50	40	0
4773/01	(DC) MEI Decision Mathematics Computation	Raw	72	46	40	34	29	24	0
		UMS	100	80	70	60	50	40	0
4776/01	(NM) MEI Numerical Methods with Coursework: Written Paper	Raw	72	62	55	49	43	36	0
4776/02	(NM) MEI Numerical Methods with Coursework: Coursework	Raw	18	14	12	10	8	7	0
4776/82	(NM) MEI Numerical Methods with Coursework: Carried Forward Coursework Mark	Raw	18	14	12	10	8	7	0
4776	(NM) MEI Numerical Methods with Coursework	UMS	100	80	70	60	50	40	0
4777/01	(NC) MEI Numerical Computation	Raw	72	55	47	39	32	25	0
		UMS	100	80	70	60	50	40	0