OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS (MEI) 4756

Further Methods for Advanced Mathematics (FP2)

Candidates answer on the Answer Booklet

OCR Supplied Materials:
- 8 page Answer Booklet
- MEI Examination Formulae and Tables (MF2)

Other Materials Required:
- Scientific or graphical calculator

Friday 11 June 2010
Morning

Duration: 1 hour 30 minutes

4756

INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer **all** the questions in Section A and **one** question from Section B.
- Do **not** write in the bar codes.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

Section A (54 marks)

Answer all the questions

1 **(a)** **(i)** Given that $\mathrm{f}(t) = \arcsin t$, write down an expression for $\mathrm{f}'(t)$ and show that

$$\mathrm{f}''(t) = \frac{t}{(1-t^2)^{\frac{3}{2}}}.$$ **[3]**

(ii) Show that the Maclaurin expansion of the function $\arcsin\left(x+\frac{1}{2}\right)$ begins

$$\frac{\pi}{6} + \frac{2}{\sqrt{3}}x,$$

and find the term in x^2. **[5]**

(b) Sketch the curve with polar equation $r = \dfrac{\pi a}{\pi + \theta}$, where $a > 0$, for $0 \leqslant \theta < 2\pi$.

Find, in terms of a, the area of the region bounded by the part of the curve for which $0 \leqslant \theta \leqslant \pi$ and the lines $\theta = 0$ and $\theta = \pi$. **[6]**

(c) Find the exact value of the integral

$$\int_0^{\frac{3}{2}} \frac{1}{9+4x^2}\,\mathrm{d}x.$$ **[5]**

2 **(a)** Given that $z = \cos\theta + \mathrm{j}\sin\theta$, express $z^n + \dfrac{1}{z^n}$ and $z^n - \dfrac{1}{z^n}$ in simplified trigonometric form.

Hence find the constants A, B, C in the identity

$$\sin^5\theta \equiv A\sin\theta + B\sin 3\theta + C\sin 5\theta.$$ **[5]**

(b) **(i)** Find the 4th roots of $-9\mathrm{j}$ in the form $r\mathrm{e}^{\mathrm{j}\theta}$, where $r > 0$ and $0 < \theta < 2\pi$. Illustrate the roots on an Argand diagram. **[6]**

(ii) Let the points representing these roots, taken in order of increasing θ, be P, Q, R, S. The mid-points of the sides of PQRS represent the 4th roots of a complex number w. Find the modulus and argument of w. Mark the point representing w on your Argand diagram. **[5]**

3 **(a)** **(i)** A 3×3 matrix **M** has characteristic equation

$$2\lambda^3 + \lambda^2 - 13\lambda + 6 = 0.$$

Show that $\lambda = 2$ is an eigenvalue of **M**. Find the other eigenvalues. **[4]**

(ii) An eigenvector corresponding to $\lambda = 2$ is $\begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}$.

Evaluate $\mathbf{M}\begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}$ and $\mathbf{M}^2\begin{pmatrix} 1 \\ -1 \\ \frac{1}{3} \end{pmatrix}$.

Solve the equation $\mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}$. **[5]**

(iii) Find constants A, B, C such that

$$\mathbf{M}^4 = A\mathbf{M}^2 + B\mathbf{M} + C\mathbf{I}.$$ **[4]**

(b) A 2×2 matrix **N** has eigenvalues -1 and 2, with eigenvectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ respectively. Find **N**. **[6]**

Section B (18 marks)

Answer one question

Option 1: Hyperbolic functions

4 **(i)** Prove, using exponential functions, that

$$\sinh 2x = 2 \sinh x \cosh x.$$

Differentiate this result to obtain a formula for $\cosh 2x$. **[4]**

(ii) Sketch the curve with equation $y = \cosh x - 1$.

The region bounded by this curve, the x-axis, and the line $x = 2$ is rotated through 2π radians about the x-axis. Find, correct to 3 decimal places, the volume generated. (You must show your working; numerical integration by calculator will receive no credit.) **[7]**

(iii) Show that the curve with equation

$$y = \cosh 2x + \sinh x$$

has exactly one stationary point.

Determine, in exact logarithmic form, the x-coordinate of the stationary point. **[7]**

Option 2: Investigation of curves

This question requires the use of a graphical calculator.

5 In parts **(i)**, **(ii)**, **(iii)** of this question you are required to investigate curves with the equation

$$x^k + y^k = 1$$

for various positive values of k.

(i) Firstly consider cases in which k is a positive even integer.

(A) State the shape of the curve when $k = 2$.

(B) Sketch, on the same axes, the curves for $k = 2$ and $k = 4$.

(C) Describe the shape that the curve tends to as k becomes very large.

(D) State the range of possible values of x and y. **[6]**

(ii) Now consider cases in which k is a positive odd integer.

(A) Explain why x and y may take any value.

(B) State the shape of the curve when $k = 1$.

(C) Sketch the curve for $k = 3$. State the equation of the asymptote of this curve.

(D) Sketch the shape that the curve tends to as k becomes very large. **[6]**

(iii) Now let $k = \frac{1}{2}$.

Sketch the curve, indicating the range of possible values of x and y. **[2]**

(iv) Now consider the modified equation $|x|^k + |y|^k = 1$.

(A) Sketch the curve for $k = \frac{1}{2}$.

(B) Investigate the shape of the curve for $k = \frac{1}{n}$ as the positive integer n becomes very large.

[4]

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GCE

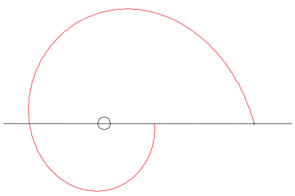
Mathematics (MEI)

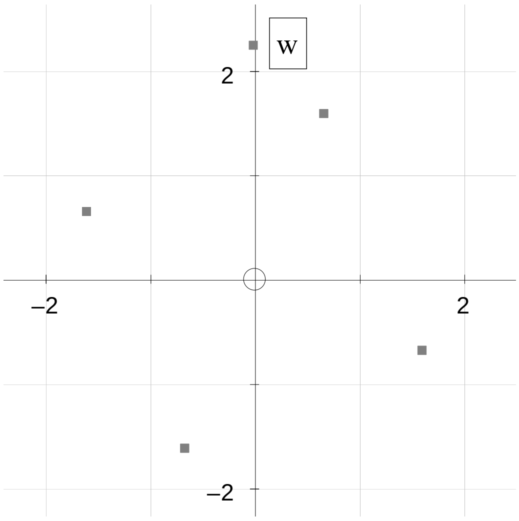
Advanced GCE **4756**

Further Methods for Advanced Mathematics (FP2)

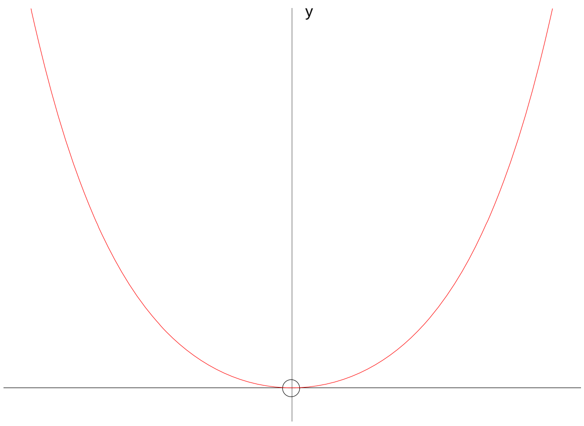
Mark Scheme for June 2010

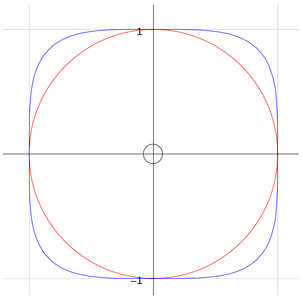
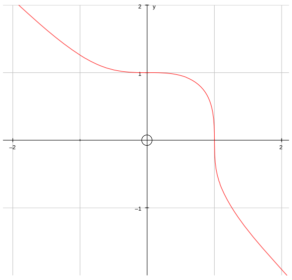
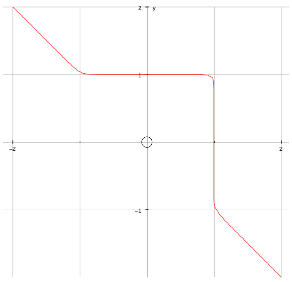
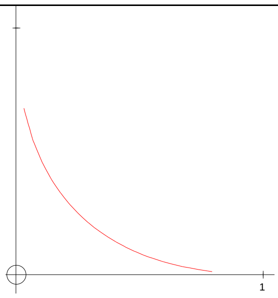
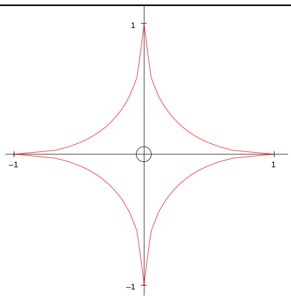
Oxford Cambridge and RSA Examinations

1 (a)(i)	$f(t) = \arcsin t$ $\Rightarrow \quad f'(t) = \dfrac{1}{\sqrt{1-t^2}} = \left(1-t^2\right)^{-\frac{1}{2}}$ $\Rightarrow \quad f''(t) = -\tfrac{1}{2}\left(1-t^2\right)^{-\frac{3}{2}} \times -2t$ $= \dfrac{t}{\left(1-t^2\right)^{\frac{3}{2}}}$	B1 M1 A1 (ag) **3**	Any form Using Chain Rule
(ii)	$f(x) = \arcsin(x + ½)$ $\Rightarrow \quad f(0) = \arcsin(½) = \dfrac{\pi}{6}$ $f'(0) = \left(1-\left(\tfrac{1}{2}\right)^2\right)^{-\frac{1}{2}} = \dfrac{2}{\sqrt{3}}$ and $\quad f''(0) = \dfrac{\frac{1}{2}}{\left(1-\left(\frac{1}{2}\right)^2\right)^{\frac{3}{2}}} = \dfrac{4\sqrt{3}}{9}$ $f(x) = f(0) + x f'(0) + \dfrac{x^2}{2} f''(0) + \ldots$ $\Rightarrow \quad$ term in x^2 is $\dfrac{2\sqrt{3}}{9}x^2$	B1 (ag) M1 A1 (ag) M1 A1 **5**	$\dfrac{\pi}{6}$ obtained clearly from $f(0)$ www Clear substitution of $x = 0$ or $t = ½$ Evaluating $f''(0)$ and dividing by 2 Accept $0.385x^2$ or better
(b)	 Area $= \displaystyle\int_0^{\pi} \tfrac{1}{2} r^2 \, d\theta$ $= \displaystyle\int_0^{\pi} \frac{\pi^2 a^2}{2(\pi+\theta)^2} d\theta = \frac{\pi^2 a^2}{2}\int_0^{\pi} \frac{1}{(\pi+\theta)^2} d\theta$ $= \dfrac{\pi^2 a^2}{2}\left[\dfrac{-1}{\pi+\theta}\right]_0^{\pi}$ $= \dfrac{\pi^2 a^2}{2}\left(\dfrac{-1}{2\pi} + \dfrac{1}{\pi}\right)$ $= \tfrac{1}{4}\pi a^2$	G1 G1 M1 A1 M1 A1 **6**	Complete spiral with $r(2\pi) < r(0)$ $r(0) = a$, $r(2\pi) = a/3$ indicated or $r(0) > r(\pi/2) > r(\pi) > r(3\pi/2) > r(2\pi)$ Dep. on G1 above Max. G1 if not fully correct Integral expression involving r^2 Correct result of integration with correct limits Substituting limits into an expression of the form $\dfrac{k}{\pi+\theta}$. Dep. on M1 above
(c)	$\displaystyle\int_0^{\frac{3}{2}} \frac{1}{9+4x^2} dx = \frac{1}{4}\int_0^{\frac{3}{2}} \frac{1}{\frac{9}{4}+x^2} dx = \frac{1}{4}\times\left[\frac{2}{3}\arctan\frac{2x}{3}\right]_0^{\frac{3}{2}}$ $= \dfrac{1}{6}\arctan 1$ $= \dfrac{\pi}{24}$	M1 A1A1 M1 A1 **5**	arctan $\dfrac{1}{4}\times\dfrac{2}{3}$ and $\dfrac{2x}{3}$ Substituting limits. Dep. on M1 above Evaluated in terms of π **19**

2 (a)	$z^n + \frac{1}{z^n} = 2\cos n\theta$, $z^n - \frac{1}{z^n} = 2j\sin n\theta$	B1	Both
	$\left(z - \frac{1}{z}\right)^5 = z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$	M1	Expanding $\left(z - \frac{1}{z}\right)^5$
	$= z^5 - \frac{1}{z^5} - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$		
	$\Rightarrow \quad 32j\sin^5\theta = 2j\sin 5\theta - 10j\sin 3\theta + 20j\sin\theta$	M1 A1	Introducing sines (and possibly cosines) of multiple angles RHS
	$\Rightarrow \quad \sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$	A1ft	Division by 32(j)
	$A = \frac{5}{8}$, $B = -\frac{5}{16}$, $C = \frac{1}{16}$	**5**	
(b)(i)	4th roots of $-9j = 9e^{\frac{3}{2}\pi j}$ are $re^{j\theta}$ where		
	$r = \sqrt{3}$	B1	Accept $9^{\frac{1}{4}}$
	$\theta = \frac{3\pi}{8}$	B1	
	$\Rightarrow \quad \theta = \frac{3\pi}{8} + \frac{2k\pi}{4}$	M1	Implied by at least two correct (ft) further values
	$\Rightarrow \quad \theta = \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$	A1	Or stating k = (0), 1, 2, 3 Allow arguments in range $-\pi \le \theta \le \pi$
		M1 A1 **6**	Points at vertices of a square centre O or 3 correct points (ft) or 1 point in each quadrant
(ii)	Mid-point of SP has argument $\frac{\pi}{8}$	B1	
	and modulus of $\sqrt{\frac{3}{2}}$	B1	
	Argument of $w = 4 \times \frac{\pi}{8} = \frac{\pi}{2}$		
	and modulus $= \left(\sqrt{\frac{3}{2}}\right)^4 = \frac{9}{4}$	M1 A1 G1 **5**	Multiplying argument by 4 and modulus raised to power of 4 Both correct w plotted on imag. axis above level of P **16**

3 (a)(i)	$2\lambda^3 + \lambda^2 - 13\lambda + 6 = 0 \Rightarrow (\lambda - 2)(2\lambda^2 + 5\lambda - 3) = 0$ $\Rightarrow \lambda = 2$ or $2\lambda^2 + 5\lambda - 3 = 0$ $\Rightarrow (2\lambda - 1)(\lambda + 3) = 0$ $\Rightarrow \lambda = \frac{1}{2}, \lambda = -3$	B1 M1 A1A1 **4**	Substituting $\lambda = 2$ or factorising Obtaining and solving a quadratic
(ii)	$\mathbf{M}\begin{pmatrix}3\\-3\\1\end{pmatrix} = 2\begin{pmatrix}3\\-3\\1\end{pmatrix} = \begin{pmatrix}6\\-6\\2\end{pmatrix}$ $\mathbf{M}^2\mathbf{v} = 2^2\mathbf{v} = 4\begin{pmatrix}1\\-1\\\frac{1}{3}\end{pmatrix} = \begin{pmatrix}4\\-4\\\frac{4}{3}\end{pmatrix}$ $\mathbf{M}\begin{pmatrix}\frac{3}{2}\\-\frac{3}{2}\\\frac{1}{2}\end{pmatrix} = 2\begin{pmatrix}\frac{3}{2}\\-\frac{3}{2}\\\frac{1}{2}\end{pmatrix} = \begin{pmatrix}3\\-3\\1\end{pmatrix}$ $\Rightarrow x = \frac{3}{2}, y = -\frac{3}{2}, z = \frac{1}{2}$	B1 B2 M1 A1 **5**	 Give B1 for one component with the wrong sign Recognising that the solution is a multiple of the given RHS Correct multiple
(iii)	$2\lambda^3 + \lambda^2 - 13\lambda + 6 = 0$ $\Rightarrow 2\mathbf{M}^3 + \mathbf{M}^2 - 13\mathbf{M} + 6\mathbf{I} = \mathbf{0}$ $\Rightarrow \mathbf{M}^3 = -\frac{1}{2}\mathbf{M}^2 + \frac{13}{2}\mathbf{M} - 3\mathbf{I}$ $\Rightarrow \mathbf{M}^4 = -\frac{1}{2}\mathbf{M}^3 + \frac{13}{2}\mathbf{M}^2 - 3\mathbf{M}$ $\Rightarrow \mathbf{M}^4 = -\frac{1}{2}(-\frac{1}{2}\mathbf{M}^2 + \frac{13}{2}\mathbf{M} - 3\mathbf{I}) + \frac{13}{2}\mathbf{M}^2 - 3\mathbf{M}$ $\Rightarrow \mathbf{M}^4 = \frac{27}{4}\mathbf{M}^2 - \frac{25}{4}\mathbf{M} + \frac{3}{2}\mathbf{I}$ $A = \frac{27}{4}, B = -\frac{25}{4}, C = \frac{3}{2}$	 M1 M1 M1 A1 **4**	 Using Cayley-Hamilton Theorem Multiplying by **M** Substituting for $\mathbf{M}^3$
(b)	$\mathbf{N} = \mathbf{PDP}^{-1}$ where $\mathbf{D} = \begin{pmatrix}-1 & 0\\0 & 2\end{pmatrix}$ and $\mathbf{P} = \begin{pmatrix}1 & -1\\2 & 1\end{pmatrix}$ $\Rightarrow \mathbf{P}^{-1} = \frac{1}{3}\begin{pmatrix}1 & 1\\-2 & 1\end{pmatrix}$ $\Rightarrow \mathbf{N} = \frac{1}{3}\begin{pmatrix}1 & -1\\2 & 1\end{pmatrix}\begin{pmatrix}-1 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\-2 & 1\end{pmatrix}$ $= \frac{1}{3}\begin{pmatrix}-1 & -2\\-2 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\-2 & 1\end{pmatrix}$ $= \frac{1}{3}\begin{pmatrix}3 & -3\\-6 & 0\end{pmatrix} = \begin{pmatrix}1 & -1\\-2 & 0\end{pmatrix}$	B1 B1 B1 B1ft M1 A1	Order must be correct For B1B1, order must be consistent Ft their **P** Attempting matrix product
	OR Let $\mathbf{N} = \begin{pmatrix}a & c\\b & d\end{pmatrix}$ $\begin{pmatrix}a & c\\b & d\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = -1\begin{pmatrix}1\\2\end{pmatrix}$ B1 $\begin{pmatrix}a & c\\b & d\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = 2\begin{pmatrix}-1\\1\end{pmatrix}$ B1 $\Rightarrow a + 2c = -1, -a + c = -2$ B1 $b + 2d = -2, -b + d = 2$ B1 $\Rightarrow a = 1, c = -1; b = -2, d = 0$ M1A1		 Or $\begin{pmatrix}a+1 & c\\b & d+1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ Or $\begin{pmatrix}a-2 & c\\b & d-2\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ Solving both pairs of equations
		6	**19**

4 (i)	2 sinh x cosh x $= 2 \times \frac{e^x + e^{-x}}{2} \times \frac{e^x - e^{-x}}{2}$ $= \frac{e^{2x} - e^{-2x}}{2}$ = sinh $2x$ Differentiating, 2 cosh $2x$ = 2 $\cosh^2 x$ + 2 $\sinh^2 x$ $\Rightarrow$ cosh $2x$ = $\cosh^2 x$ + $\sinh^2 x$	M1 A1 (ag) B1 B1 **4**	Using exponential definitions and multiplying or factorising One side correct Correct completion
(ii)	 Volume $= \pi \int_0^2 (\cosh x - 1)^2 \, dx$ $= \pi \int_0^2 \cosh^2 x - 2\cosh x + 1 \, dx$ $= \pi \int_0^2 \frac{1}{2}\cosh 2x - 2\cosh x + \frac{3}{2} \, dx$ $= \pi \left[\frac{1}{4}\sinh 2x - 2\sinh x + \frac{3}{2}x \right]_0^2$ $= \pi \left[\frac{1}{4}\sinh 4 - 2\sinh 2 + 3 \right]$ = 8.070	G1 M1 A1 M1 A2 A1 **7**	Correct shape and through origin $\int (\cosh x - 1)^2 \, dx$ A correct expanded integral expression including limits 0, 2 (may be implied by later work) Attempting to obtain an integrable form Dep. on M1 above Give A1 for two terms correct 3 d.p. required. Condone 8.07
(iii)	y = cosh $2x$ + sinh x $\Rightarrow \frac{dy}{dx}$ = 2 sinh $2x$ + cosh x At S.P. 2 sinh $2x$ + cosh x = 0 $\Rightarrow$ 4 sinh x cosh x + cosh x = 0 $\Rightarrow$ cosh x(4 sinh x + 1) = 0 $\Rightarrow$ cosh x = 0 (rejected) $\Rightarrow$ sinh $x = -\frac{1}{4}$ $\Rightarrow x = \ln\left(-\frac{1}{4} + \frac{\sqrt{17}}{4}\right)$	B1 M1 M1 A1 A1 M1 A1 **7**	Any correct form Setting derivative equal to zero and using identity Solving $\frac{dy}{dx}$=0 to obtain value of sinh x Repudiating cosh x = 0 Using log form of arsinh, or setting up and solving quadratic in e^x A0 if extra "roots" quoted **18**

5(i)(A)	Circle	B1	
(B)		G1 G1	Sketch of circle, centre (0, 0) Sketch of "squarer" circle on same axes
(C)	Square	B1	
(D)	$-1 \le x \le 1$	B1	Give B1B0 for not all non-strict or unclear
	$-1 \le y \le 1$	B1	
		6	
(ii)(A)	Odd roots exist for all real numbers	B1	Any equivalent explanation
(B)	Line	B1	Sketch insufficient
(C)		G1	
	Asymptote: $x + y = 0$	B1	
(D)		G1 G1	Line $x + y = 0$ outside unit square Lines $y = 1$ and $x = 1$ on unit square
		6	
(iii)		G1	G0 if curve beyond (1, 0) or (0, 1)
	$0 \le x, y \le 1$	B1	Accept strict, or indication on graph
		2	
(iv)(A)		G2ft	Give G1 for a partial attempt. Ft from (iii) on shape
(B)	Limit is a "plus sign"	B1	
	where $x \to 0$ for $-1 \le y \le 1$ and vice versa	B1	
		4	**18**