OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS (MEI) **4758/01**

Differential Equations

Candidates answer on the Answer Booklet

OCR Supplied Materials:
- 8 page Answer Booklet
- Graph paper
- MEI Examination Formulae and Tables (MF2)

Other Materials Required:
None

Wednesday 21 January 2009
Afternoon

Duration: 1 hour 30 minutes

4758 01

INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer any **three** questions.
- Do **not** write in the bar codes.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

1 The differential equation

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 2$$

is to be solved.

(i) Write down the auxiliary equation. Show that -2 is a root of this equation and find the other two roots. Hence write down the complementary function. **[6]**

(ii) Find the general solution. **[3]**

When $x = 0$, $y = 0$ and when $x = \ln 2$, $y = 0$. As $x \to \infty$, y tends to a finite limit.

(iii) Show that $y = -2e^{-2x} + 3e^{-x} - 1$. **[6]**

(iv) Show that $y = 0$ only when $x = 0$ or $\ln 2$. Show also that the graph of y against x has only one stationary point, and determine its coordinates. **[5]**

(v) Sketch the graph of the solution for $x \geqslant 0$. **[4]**

2 The differential equation

$$\frac{dy}{dx}\cos x + y\sin x = x\cos^2 x$$

is to be solved for $|x| < \frac{1}{2}\pi$ subject to the condition that $y = 1$ when $x = 0$.

(i) Find the solution. **[10]**

(ii) Sketch the solution curve. **[2]**

Now consider the differential equation

$$\frac{dy}{dx}\cos x + y\sin x = x\cos x\sin x$$

for $|x| < \frac{1}{2}\pi$, subject to the condition that $y = 1$ when $x = 0$.

(iii) Use Euler's method with a step length of 0.1 to estimate y when $x = 0.2$. The algorithm is given by $x_{r+1} = x_r + h$, $y_{r+1} = y_r + hy'_r$. **[6]**

(iv) Use the integrating factor method and the numerical approximation

$$\int_0^{0.2} x\tan x\,dx \approx 0.002\,688$$

to estimate y when $x = 0.2$. **[6]**

3 An oil drum of mass 60 kg is dropped from rest from a point A which is at a height of 10 m above a lake. The oil drum is modelled as a particle that moves vertically. When it is x m below A, its speed is $v\,\mathrm{m\,s^{-1}}$. Before it enters the water, the forces acting on it are its weight and a resistance force of magnitude $\frac{1}{4}v^2$ N.

(i) Show that

$$\frac{v}{240g - v^2}\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{240}$$

and hence find v^2 in terms of x. **[9]**

(ii) Show that the speed of the oil drum as it reaches the water is $13.71\,\mathrm{m\,s^{-1}}$, correct to two decimal places. **[1]**

After it enters the water, the forces acting on the oil drum are its weight, a resistance force of magnitude $60v$ N and a buoyancy force of $90g$ N vertically upwards.

Assume that the initial speed in the water is $13.71\,\mathrm{m\,s^{-1}}$ and that the oil drum moves vertically.

(iii) Show that t seconds after entering the water its speed is given by $v = 18.61\mathrm{e}^{-t} - 4.9$. **[8]**

(iv) Calculate the greatest depth below the surface of the water that the oil drum reaches. **[6]**

4 The simultaneous differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -3x - y + 7$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2x - y + 2$$

are to be solved for $t \geqslant 0$.

(i) Find the values of x and y for which $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t} = 0$. **[2]**

(ii) Show that

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + 5x = 5.$$ **[5]**

(iii) Find the general solution for x. **[6]**

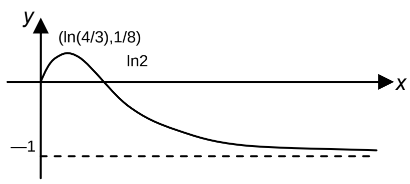
(iv) Find the corresponding general solution for y. **[3]**

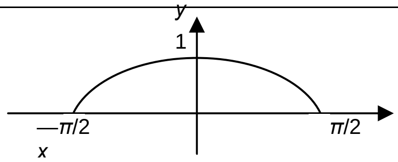
When $t = 0$, $x = 4$ and $y = 0$.

(v) Find the solutions for x and y. **[3]**

(vi) Sketch the graphs of x against t and y against t, for $t \geqslant 0$. Explain how your solution to part **(i)** relates to your graphs. **[5]**

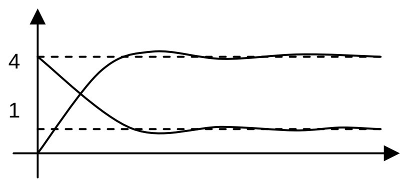
4758 Differential Equations

1(i)	$\alpha^3 + 2\alpha^2 - \alpha - 2 = 0$	B1		
	$(-2)^3 + 2(-2)^2 - (-2) - 2 = 0$	E1	Or factorise	
	$(\alpha + 2)(\alpha^2 - 1) = 0$	M1	Solve	
	$\alpha = -2, \pm 1$	A1		
	$y = Ae^{-2x} + Be^{-x} + Ce^{x}$	M1	Attempt CF	
		F1	CF for their three roots	
				6
(ii)	PI $y = \frac{2}{-2} = -1$	M1	Constant PI	
		A1	Correct PI	
	GS $y = -1 + Ae^{-2x} + Be^{-x} + Ce^{x}$	F1	GS = PI + CF	
				3
(iii)	$e^x \to \infty$ as $x \to \infty$	M1	Consider as $x \to \infty$	
	so finite limit $\Rightarrow C = 0$	F1	Must be shown, not just stated	
	$x = 0$, y $= 0 \Rightarrow 0 = -1 + A + B$	M1	Use condition	
	$x = \ln 2, y = 0 \Rightarrow 0 = -1 + \frac{1}{4}A + \frac{1}{2}B$	M1	Use condition	
	Solving gives A = -2, B = 3	M1		
	$y = -2e^{-2x} + 3e^{-x} - 1$	E1	Convincingly shown	
				6
(iv)	$y = -(2e^{-x} - 1)(e^{-x} - 1)$			
	$y = 0 \Leftrightarrow e^{-x} = \frac{1}{2}$ or 1	M1	Solve	
	$\Leftrightarrow x = \ln 2$ or 0	E1	Convincingly show no other roots	
	$\frac{dy}{dx} = 4e^{-2x} - 3e^{-x} = e^{-x}(4e^{-x} - 3)$			
	$\frac{dy}{dx} = 0 \Leftrightarrow e^{-x} = \frac{3}{4}$ as $e^{-x} \neq 0$	M1	Solve	
	$\Leftrightarrow x = \ln \frac{4}{3}$	E1	Show only one root	
	Stationary point at $(\ln \frac{4}{3}, \frac{1}{8})$	A1		
				5
(v)		B1	Through (0, 0)	
		B1	Through (ln 2, 0)	
		B1	Stationary point at their answer to (iv)	
		B1	$y \to -1$ as $x \to \infty$	
				4

2(i)	$\frac{dy}{dx} + y\tan x = x\cos x$	M1	Rearrange	
	$I = \exp\int \tan x dx$	M1	Attempt IF	
	$= \exp \ln \sec x$	A1	Correct IF	
	$= \sec x$	A1	Simplified	
	$\frac{d}{dx}(y\sec x) = x$	M1	Multiply and recognise derivative	
	$y\sec x = \frac{1}{2}x^2 + A$	M1	Integrate	
		A1	RHS	
	$y = (\frac{1}{2}x^2 + A)\cos x$	F1	Divide by their IF (must divide constant)	
	$x = 0, y = 1 \Rightarrow A = 1$	M1	Use condition	
	$y = (\frac{1}{2}x^2 + 1)\cos x$	F1	Follow their non-trivial GS	
				10
(ii)		B1	Shape correct for $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$	
		B1	Through (0,1)	
				2
(iii)	$y' = \frac{x\cos x\sin x - y\sin x}{\cos x}$	M1	Rearrange	
	$y'(0) = 0$	B1		
	$y(0.1) = 1$	B1		
	$y'(0.1) = -0.090351$	B1		
	$y(0.2) = 1 + 0.1 \times -0.090351 = 0.990965$	M1	Use of algorithm for second step	
		A1	3sf or better	
				6
(iv)	$I = \sec x$	M1	Same IF as in (i) or attempt from scratch	
	$\frac{d}{dx}(y\sec x) = x\tan x$	A1		
	$[y\sec x]_{x=0}^{x=0.2} = \int_0^{0.2} x\tan x dx$	M1	Integrate	
		A1	Accept no limits	
	$y(0.2)\sec(0.2) - 1 \times \sec 0 \approx 0.002688$	M1	Substitute limits (both sides)	
	$\Rightarrow y(0.2) \approx 0.982701$	A1	Awrt 0.983	
				6

3(i)	$60v\frac{dv}{dx} = 60g - \frac{1}{4}v^2$	M1	N2L	
		A1	Correct N2L equation	
	$\frac{v}{240g - v^2}\frac{dv}{dx} = \frac{1}{240}$	E1	Convincingly shown	
	$\int \frac{v}{240g - v^2}dv = \int \frac{1}{240}dx$	M1	Integrate	
	$-\frac{1}{2}\ln\lvert 240g - v^2\rvert = \frac{1}{240}x + c$	A1	$\ln\lvert 240g - v^2\rvert$ seen	
		A1	RHS	
	$240g - v^2 = Ae^{\frac{x}{120}}$	M1	Rearrange, dealing properly with constant	
	$x = 0, v = 0 \Rightarrow A = 240g$	M1	Use condition	
	$v^2 = 240g(1 - e^{-\frac{x}{120}})$	A1	Cao	9
(ii)	$x = 10 \Rightarrow v = \sqrt{240g(1 - e^{-\frac{10}{120}})} \approx 13.71$	E1	Convincingly shown	1
(iii)	$60\frac{dv}{dt} = 60g - 60v - 90g$	M1	N2L	
	$\frac{dv}{dt} = -\frac{1}{2}g - v$ or $\frac{dv}{dt} + v = -\frac{1}{2}g$	A1	Correct DE	
	Solving DE (three alternative methods):			
	$\int \frac{dv}{v + \frac{1}{2}g} = \int -dt$	M1	Separate	
	$\ln\lvert v + \frac{1}{2}g\rvert = -t + k$	M1	Integrate	
		A1	LHS	
	$v + \frac{1}{2}g = Ae^{-t}$	M1	Rearrange, dealing properly with constant	
	or			
	$\alpha + 1 = 0 \Rightarrow \alpha = -1$		*M1 Solve auxiliary equation*	
	CF Ae^{-t}		*M1 CF for their root*	
	PI $-\frac{1}{2}g$		*M1 Attempt to find constant PI*	
	$v = Ae^{-t} - \frac{1}{2}g$		*A1 All correct*	
	or			
	$I = e^t$		*M1 Attempt integrating factor*	
	$\frac{d}{dt}(e^t v) = -\frac{1}{2}ge^t$		*M1 Multiply*	
	$e^t v = -\frac{1}{2}ge^t + A$		*M1 Integrate*	
	$v = Ae^{-t} - \frac{1}{2}g$		*A1 All correct*	
	$v = 13.71, t = 0 \Rightarrow 13.71 = A - \frac{1}{2}g \Rightarrow A = 18.61$	M1	Use condition	
	$v = 18.61e^{-t} - 4.9$	E1	Complete argument	8

(iv)	At greatest depth, $v = 0$	M1	Set velocity to zero and attempt to solve	
	$\Rightarrow e^{-t} = \dfrac{4.9}{18.61} \Rightarrow t = 1.3345$	A1		
	Depth $= \int_0^{1.3345} (18.61e^{-t} - 4.9)dt$	M1	Integrate	
	$= \left[-18.61e^{-t} - 4.9t\right]_0^{1.3345}$	A1	Ignore limits	
		M1	Use limits (or evaluate constant and substitute for t)	
	= 7.17 m	A1	All correct	6

4(i)	$\left.\begin{array}{r} -3x - y + 7 = 0 \\ 2x - y + 2 = 0 \end{array}\right\} \Leftrightarrow \begin{array}{l} x = 1 \\ y = 4 \end{array}$	B1 B1		2
(ii)	$\ddot{x} = -3\dot{x} - \dot{y}$	M1	Differentiate	
	$= -3\dot{x} - (2x - y + 2)$	M1	Substitute for $\dot{y}$	
	$y = -3x + 7 - \dot{x}$	M1	y in terms of $x, \dot{x}$	
	$\ddot{x} = -3\dot{x} - 2x - 3x + 7 - \dot{x} - 2$	M1	Substitute for y	
	$\Rightarrow \ddot{x} + 4\dot{x} + 5x = 5$	E1	Complete argument	5
(iii)	$\alpha^2 + 4\alpha + 5 = 0$	M1	Auxiliary equation	
	$\Rightarrow \alpha = -2 \pm i$	A1		
		M1	CF for complex roots	
	CF $e^{-2t}(A\cos t + B\sin t)$	F1	CF for their roots	
	PI $x = \dfrac{5}{5} = 1$	B1		
	GS $x = 1 + e^{-2t}(A\cos t + B\sin t)$	F1	GS = PI + CF with two arbitrary constants	6
(iv)	$y = -3x + 7 - \dot{x}$	M1	y in terms of $x, \dot{x}$	
	$\dot{x} = -2e^{-2t}(A\cos t + B\sin t) + e^{-2t}(-A\sin t + B\cos t)$	M1	Differentiate their x (product rule)	
	$y = 4 + e^{-2t}((A - B)\sin t - (A + B)\cos t)$	A1	Constants must correspond	3
(v)	$1 + A = 4$	M1	Use condition on x	
	$4 - A - B = 0$	M1	Use condition on y	
	$A = 3, B = 1$			
	$x = 1 + e^{-2t}(3\cos t + \sin t)$			
	$y = 4 + e^{-2t}(2\sin t - 4\cos t)$	A1	Both solutions	3
(vi)		B1 B1 B1 B1	(0, 4) → 1 (0, 0) → 4	
	As the solutions approach the asymptotes, the gradients approach zero.	B1	Must refer to gradients	5

4758 Differential Equations (Written Examination)

General Comments

Many candidates demonstrated a good understanding of the specification and high levels of algebraic competency.

When sketching graphs, candidates are not expected to do any detailed analysis, but they should identify features which have been given or obtained in the question.

Comments on Individual Questions

1) (i) This was generally done very well, although some candidates made errors in the other roots of the auxiliary equation.

(ii) Most candidates correctly identified the particular integral. Some unnecessarily tried a linear or quadratic expression; this often resulted in a correct answer but was more time consuming and error-prone.

(iii) Many candidates stated one of the arbitrary constants is zero without explanation. For a given answer some explanation must be given. Most candidates successfully calculated the other constants.

(iv) Many candidates found the coordinates of the stationary point. Some were unable to justify that there are only two roots, often wrongly assuming it was sufficient to substitute the values of x and show $y = 0$.

(v) The sketch was often broadly correct, but sometimes did not label the relevant details – the given conditions and the calculated stationary point.

2) (i) This was often done well, but some candidates made a sign error leading to an integrating factor of cos x rather than sec x. Candidates are advised to check their integrating factor carefully to avoid such an error.

(ii) The sketches were often poor. Detailed curve sketching was not required, but candidates were expected to indicate the initial condition and give some indication of scale on the x-axis.

(iii) The numerical solution was generally well done. However some candidates gave a list of incorrect numbers with no evidence of method for which no credit can be given.

(iv) Some candidates struggled to make any progress. Many made some progress by using the correct integrating factor. However, few completely correct answers were seen as candidates often failed to deal correctly with the initial condition.

3) (i) Many candidates were able to write down a correct equation of motion, but some did not realise that the acceleration can be expressed as $v\frac{dv}{dx}$. Most could solve the differential equation, although a few candidates omitted the constant. It is vital for candidates to realise the importance of including the constant and dealing properly with it when rearranging their solution.

(ii) Candidates who solved the differential equation correctly almost always completed this calculation correctly.

(iii) Most candidates were able to write down a correct equation of motion and solve it, although some again omitted the constant of integration.

(iv) Most candidates integrated the velocity but some omitted the constant of integration or made errors in calculating the constant.

4) (i) Almost all candidates correctly calculated the values of x and y.

(ii) The vast majority of candidates took a correct approach to the elimination of y but some made algebraic errors in the process.

(iii) Most candidates correctly found the general solution, but some made errors in the roots of the auxiliary equation.

(iv) It was pleasing to see the vast majority of candidates using their general solution and the first differential equation to find y. To find the *corresponding* solution, candidates should not construct and solve a new differential equation and only a few candidates used this incorrect approach.

(v) Generally candidates were able to use the given conditions to find the solutions.

(vi) The sketches were often done well, but some candidates omitted to identify the key features, in particular the initial conditions and the asymptotes.