OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS **2605**

Pure Mathematics 5

Tuesday **28 JUNE 2005** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 3 printed pages and 1 blank page.

[Turn over

1 The equation $x^3 + 3x^2 + 8x + 10 = 0$ has roots α, β and γ.

(i) Write down the values of $\alpha + \beta + \gamma$, $\beta\gamma + \gamma\alpha + \alpha\beta$ and $\alpha\beta\gamma$. [3]

(ii) Find $\alpha^2 + \beta^2 + \gamma^2$. [2]

(iii) Describe the nature of the roots. [2]

(iv) Show that $\alpha^2\beta + \alpha^2\gamma + \beta^2\gamma + \beta^2\alpha + \gamma^2\alpha + \gamma^2\beta = 6$. [4]

(v) Show that $\beta\gamma(\gamma + \alpha)(\alpha + \beta) = 8\beta\gamma - 10\alpha$, and deduce the value of

$$\beta\gamma(\gamma + \alpha)(\alpha + \beta) + \gamma\alpha(\alpha + \beta)(\beta + \gamma) + \alpha\beta(\beta + \gamma)(\gamma + \alpha).$$ [4]

(vi) Find a cubic equation with integer coefficients which has roots

$$\alpha(\beta + \gamma), \beta(\gamma + \alpha) \text{ and } \gamma(\alpha + \beta).$$ [5]

2 **(a)** **(i)** Given that $c \geqslant 1$ and $\cosh x = c$, show that $x = \pm\ln\left(c + \sqrt{c^2 - 1}\right)$. [6]

(ii) Solve the equation $\sinh^2 x + 3\cosh x = 9$, giving the answers in an exact logarithmic form. [5]

(b) Let $f(x) = \arcsin\left(\frac{3}{5} + x\right)$.

(i) Find $f'(x)$ and $f''(x)$. [3]

(ii) Given that the Maclaurin series for $f(x)$ begins

$$\arcsin\tfrac{3}{5} + px + qx^2 + \dots,$$

find p and q. [3]

(iii) Use this Maclaurin series to calculate an approximate value of $\int_0^{0.1} f(x)\,dx$, giving your answer to 4 decimal places. [3]

3 **(i)** Express in a simplified trigonometric form

(A) $e^{j\theta} + e^{-j\theta}$,

(B) $(1 - 3e^{j\theta})(1 - 3e^{-j\theta})$. [5]

(ii) Series C and S are defined by

$$C = \cos\theta + 3\cos 2\theta + 9\cos 3\theta + \ldots + 3^{n-1}\cos n\theta,$$

$$S = \sin\theta + 3\sin 2\theta + 9\sin 3\theta + \ldots + 3^{n-1}\sin n\theta.$$

Show that $C = \dfrac{\cos\theta + 3^{n+1}\cos n\theta - 3^n\cos(n+1)\theta - 3}{10 - 6\cos\theta}$,

and find a similar expression for S. [8]

(iii) Find the three cube roots of 27j, in the form $re^{j\theta}$ where $r > 0$ and $-\pi < \theta \leqslant \pi$. [3]

(iv) Given that $w^3 = 27j$, find the possible values of $(1 - w)(1 - w^*)$. [4]

4 **(a)** The curve C has polar equation $r = a(5 - 4\sin\theta)$, where a is a positive constant.

(i) For points on C, show that the maximum value of $r\sin\theta$ is $\frac{25}{16}a$. [5]

(ii) Sketch the curve C. [3]

(iii) Find the area of the region enclosed by the lines $\theta = 0$, $\theta = \frac{1}{2}\pi$ and the arc of C for which $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. [6]

(b) A conic has polar equation $\dfrac{6k}{r} = 7 + 5\cos\theta$, where k is a positive constant.

(i) Sketch the conic. [2]

(ii) The origin O is a focus of the conic. Give the polar coordinates of the other focus S. [2]

(iii) Given that P is a point on the conic, find the total distance OP + PS. [2]

Mark Scheme 2605
June 2005

1 (i)	$\sum\alpha = -3,\quad \sum\alpha\beta = 8,\quad \alpha\beta\gamma = -10$	B1B1B1 **3**	
(ii)	$\sum\alpha^2 = \left(\sum\alpha\right)^2 - 2\sum\alpha\beta$ $= (-3)^2 - 2(8)$ $= -7$	M1 A1 **2**	For correct formula
(iii)	Roots cannot all be real One real, two complex (conjugate) roots	M1 A1 **2**	*Or* there are complex roots *Accept 'imaginary' for 'complex'* *When* $\sum\alpha^2 > 0$ *, M1 cannot be awarded, but give B2 for 'one real and two complex'*
(iv)	$\sum\alpha^2\beta = \left(\sum\alpha\right)\left(\sum\alpha\beta\right) - 3\alpha\beta\gamma$ $= (-3)(8) - 3(-10)$ $= 6$	M1A1 M1 A1 (ag) **4**	or $\sum\alpha^2\beta = \left(\sum\alpha\right)\left(\sum\alpha^2\right) - \sum\alpha^3$ *and* $\sum\alpha^3 + 3\sum\alpha^2 + 8\sum\alpha + 30 = 0$ *Dependent on previous M1*
(v)	$\beta\gamma(\gamma\alpha + \gamma\beta + \alpha^2 + \alpha\beta) = \beta\gamma(8 + \alpha^2)$ $= 8\beta\gamma + \alpha(\alpha\beta\gamma)$ $= 8\beta\gamma - 10\alpha$ $\sum\beta\gamma(\gamma + \alpha)(\alpha + \beta) = 8\sum\alpha\beta - 10\sum\alpha$ $= 8(8) - 10(-3)$ $= 94$	M1 A1 (ag) M1 A1 **4**	
(vi)	Sum $= 2\sum\alpha\beta \quad (=16)$ Sum in pairs = 94 Product $= \alpha\beta\gamma\left(\sum\alpha^2\beta + 2\alpha\beta\gamma\right) \quad (=140)$ Equation is $y^3 - 16y^2 + 94y - 140 = 0$	B1 M1A1 M1 A1 **5**	 Forming cubic equation (with numerical coefficients) A0 if =0 omitted

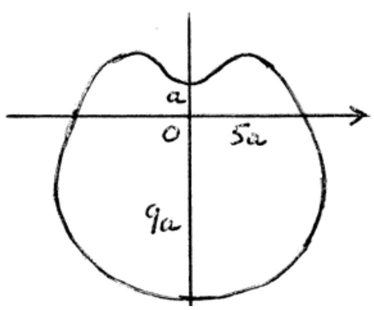
	OR Let $y = 8 + \frac{10}{x}$, $x = \frac{10}{y-8}$ M1		
	$\left(\frac{10}{y-8}\right)^3 + 3\left(\frac{10}{y-8}\right)^2 + 8\left(\frac{10}{y-8}\right) + 10 = 0$ A1		
	$1000 + 300(y-8) + 80(y^2 - 16y + 64)$ $+ 10(y^3 - 24y^2 + 192y - 512) = 0$ M1		
	$y^3 - 16y^2 + 94y - 140 = 0$ A2		Give A1 if just one slip made

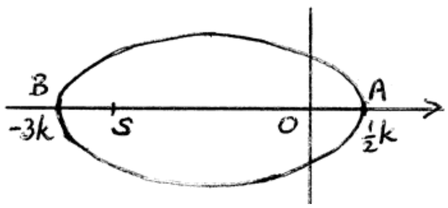
2(a)(i)	$\frac{1}{2}(e^x + e^{-x}) = c$ $e^{2x} - 2ce^x + 1 = 0$ $e^x = \frac{2c \pm \sqrt{4c^2 - 4}}{2}$ $= c \pm \sqrt{c^2 - 1}$ $x = \ln(c \pm \sqrt{c^2 - 1})$ $(c + \sqrt{c^2 - 1})(c - \sqrt{c^2 - 1}) = c^2 - (c^2 - 1) = 1$ Hence $\ln(c - \sqrt{c^2 - 1}) = \ln(c + \sqrt{c^2 - 1})^{-1}$ $= -\ln(c + \sqrt{c^2 - 1})$	M1 M1 M1 A1 M1 A1 **6**	or M3 for a complete alternative method
(ii)	$\cosh^2 x - 1 + 3\cosh x = 9$ $\cosh^2 x + 3\cosh x - 10 = 0$ $(\cosh x - 2)(\cosh x + 5) = 0$ $\cosh x = 2$ $x = \pm\ln(2 + \sqrt{3})$	M1 M1 A1 M1A1 **5**	Using $\cosh^2 x - \sinh^2 x = 1$ *(using wrong identity is M0)* Solving quadratic *Or* $\ln(2 \pm \sqrt{3})$ (A0 if any other solutions given)
	OR Writing in exponential form and obtaining quadratic factors M2 $(e^{2x} - 4e^x + 1)(e^{2x} + 10e^x + 1) = 0$ A1 $e^x = 2 \pm \sqrt{3}$ M1 $x = \ln(2 \pm \sqrt{3})$ A1		for $(e^{2x} - 4e^x + 1)$ Obtaining a value for *x* in log form
(b)(i)	$f'(x) = \frac{1}{\sqrt{1 - (\frac{3}{5} + x)^2}}$ $f''(x) = \left(\frac{3}{5} + x\right)\left(1 - (\frac{3}{5} + x)^2\right)^{-\frac{3}{2}}$	B1 M1A1 **3**	

(ii)	$f(0)=\arcsin\frac{3}{5}$ $f'(0)=\dfrac{5}{4},\quad f''(0)=\dfrac{75}{64}$ $f(0)+x\,f'(0)+\dfrac{x^2}{2!}f''(0)+\ldots$ $=\arcsin\dfrac{3}{5}+\dfrac{5}{4}x+\dfrac{75}{128}x^2+\ldots$	M1 A1A1 ft **3**	Evaluating $f'(0)$ *or* $f''(0)$ For $p=\dfrac{5}{4}$ and $q=\dfrac{75}{128}$ *(ft requires non-zero values)*
(iii)	$\left[x\arcsin\dfrac{3}{5}+\dfrac{5}{8}x^2+\dfrac{25}{128}x^3+\ldots\right]_0^{0.1}$ $=0.064350+0.00625+0.000195+\ldots$ $=0.0708$	B1 ft M1 A1 **3**	ft requires three non-zero terms Evaluating three non-zero terms

3 (i)	$(A)\quad(\cos\theta+\mathrm{j}\sin\theta)+(\cos\theta-\mathrm{j}\sin\theta)$ $=2\cos\theta$ $(B)\quad 1-3(\mathrm{e}^{\mathrm{j}\theta}+\mathrm{e}^{-\mathrm{j}\theta})+9$ $=10-6\cos\theta$	M1 A1 M1A1 A1 **5**	For $\mathrm{e}^{\mathrm{j}\theta}=\cos\theta+\mathrm{j}\sin\theta$ or $(1-3\cos\theta)^2+9\sin^2\theta$
(ii)	$C+\mathrm{j}S=\mathrm{e}^{\mathrm{j}\theta}+3\mathrm{e}^{2\mathrm{j}\theta}+9\mathrm{e}^{3\mathrm{j}\theta}+\ldots$ $=\dfrac{\mathrm{e}^{\mathrm{j}\theta}(1-[3\mathrm{e}^{\mathrm{j}\theta}]^n)}{1-3\mathrm{e}^{\mathrm{j}\theta}}$ $=\dfrac{\mathrm{e}^{\mathrm{j}\theta}(1-3^n\mathrm{e}^{\mathrm{j}n\theta})(1-3\mathrm{e}^{-\mathrm{j}\theta})}{(1-3\mathrm{e}^{\mathrm{j}\theta})(1-3\mathrm{e}^{-\mathrm{j}\theta})}$ $=\dfrac{\mathrm{e}^{\mathrm{j}\theta}-3+3^{n+1}\mathrm{e}^{\mathrm{j}n\theta}-3^n\mathrm{e}^{\mathrm{j}(n+1)\theta}}{10-6\cos\theta}$ $C=\dfrac{\cos\theta-3+3^{n+1}\cos n\theta-3^n\cos(n+1)\theta}{10-6\cos\theta}$ $S=\dfrac{\sin\theta+3^{n+1}\sin n\theta-3^n\sin(n+1)\theta}{10-6\cos\theta}$	M1 M1 A1 M1 A1 M1 A1 (ag) A1 **8**	Obtaining a geometric series Summing a geometric series Using conjugate of denominator Expression with real denominator and numerator multiplied out Equating real or imaginary parts Correctly obtained *Summing to infinity can earn all the M marks but no A marks*
(iii)	$3\mathrm{e}^{\mathrm{j}\frac{\pi}{6}},\quad 3\mathrm{e}^{\mathrm{j}\frac{5\pi}{6}},\quad 3\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}$	B1B1B1 **3**	If B0, give B2 for 3 arguments correct B1 for 2 arguments correct

(iv)	$w = 3e^{j\theta}$ where $\theta = \frac{\pi}{6},\ \frac{5\pi}{6},\ -\frac{\pi}{2}$ $(1-w)(1-w^*) = (1-3e^{j\theta})(1-3e^{-j\theta})$ $= 10 - 6\cos\theta$ $= 10 - 3\sqrt{3},\ 10 + 3\sqrt{3},\ 10$	M1 B1B1B1 **4**	Implied if next B3 earned Accept 4.8, 15.2

4(a)(i)	$y = r\sin\theta = a(5\sin\theta - 4\sin^2\theta)$ $\frac{dy}{d\theta} = a(5\cos\theta - 8\sin\theta\cos\theta)$ $= 0$ when $\cos\theta = 0$ or $\sin\theta = \frac{5}{8}$ When $\sin\theta = \frac{5}{8}$, maximum $y = a(5 \times \frac{5}{8} - 4 \times \frac{25}{64})$ $= \frac{25}{16}a$	M1 A1 M1 A1 A1 (ag) **5**	Differentiating $r\sin\theta$ Solving $\frac{dy}{d\theta} = 0$ For $\sin\theta = \frac{5}{8}$
	OR $y = r\sin\theta = a(5\sin\theta - 4\sin^2\theta)$ M1 $= a\left[\frac{25}{16} - \left(2\sin\theta - \frac{5}{4}\right)^2\right]$ A1A1 $\leq \frac{25}{16}a$ M1A1		Completing the square
(ii)		B1 B1 B1 **3**	Correct shape in 1st or 2nd quadrant Correct shape in 3rd or 4th quadrant Fully correct, with a, $5a$, $9a$ shown, and zero gradient when crossing the y-axis

(iii)	Area $= \int_0^{\frac{1}{2}\pi} \frac{1}{2}a^2(5-4\sin\theta)^2\,d\theta$ $= \int_0^{\frac{1}{2}\pi} \frac{1}{2}a^2(25-40\sin\theta+8-8\cos 2\theta)\,d\theta$ $= \frac{1}{2}a^2\Big[\ 33\theta+40\cos\theta-4\sin 2\theta\ \Big]_0^{\frac{1}{2}\pi}$ $= \frac{1}{4}a^2(33\pi-80)$	M1 A1 B1 B1B1 ft B1 **6**	Integral of r^2 Correct integral expression For $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ Integrating $a+b\sin\theta$ and $c\cos 2\theta$ Accept $5.92a^2$
(b)(i)		B1 B1 **2**	For any ellipse Ellipse with O as RH focus
(ii)	BS = OA S is $\left(\frac{5}{2}k,\ \pi\right)$	M1 A1 **2**	
(iii)	OP + PS = length of major axis $= \frac{7}{2}k$	M1 A1 **2**	

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General Comments

There was a wide range of performance on this paper, with about a quarter of the candidates scoring 50 marks or more (out of 60), and about a quarter scoring less than 30 marks. Almost every candidate answered questions 1 and 2; then about 80% chose question 3 and only 20% chose question 4.

Comments on Individual Questions

1) **Roots of a cubic equation**

This was by far the best answered question, with half the attempts scoring 17 marks or more (out of 20). For many candidates this question provided a high proportion of their total mark.

Parts (i) and (ii) were almost always answered correctly.

In part (iii), most candidates mentioned the existence of complex roots; but relatively few earned both marks by stating that one root is real and two are complex.

Parts (iv) and (v) were very often answered efficiently and correctly, although some candidates set off on the wrong algebraic track and wasted a lot of time in fruitless effort.

Finding the new cubic equation in part (vi) was well understood, and the product of the new roots was very often found correctly. Many candidates did not realise that they had already found the sum of products in pairs, and calculated this again, often obtaining a value different from their answer to part (v).

2) The average mark on this question was about 13.

(a) **Hyperbolic functions**

In part (i), most candidates were able to show that $x = \ln(c \pm \sqrt{c^2 - 1})$, but only a few then showed correctly that this is equivalent to the desired result $x = \pm\ln(c + \sqrt{c^2 - 1})$.

In part (ii), those who used $\sinh^2 x = \cosh^2 x - 1$ were usually able to obtain $\cosh x = 2$ and hence write x in logarithmic form, but the other solution $\cosh x = -5$ was sometimes not rejected. Those who wrote the original equation in exponential form very rarely made any progress.

(b) **Inverse circular functions and Maclaurin series**

In part (i), the double differentiation of $\arcsin(\frac{3}{5}+x)$ caused a surprising number of problems, notably sign errors.
In part (ii), the Maclaurin series usually followed correctly from the results in part (i), although many forgot to divide $f''(0)$ by 2 when finding q.
Most candidates knew what to do in part (iii), but $0.1\arcsin(0.6)$ was often evaluated as $\arcsin(0.06)$, and degrees were sometimes used instead of radians.

3) **Complex numbers**

The average mark on this question was about 11.
Part (i) was generally answered well, but the responses to part (ii) ranged quite uniformly from very poor to fully correct. Most candidates began by considering $C+\mathrm{j}S$, but some made no progress beyond this. A common stumbling block, when the geometric series had been summed, was the failure to make the denominator real. Careless errors such as $(3e^{j\theta})^n = 3e^{jn\theta}$, and sign errors, spoilt some otherwise good attempts, and the expression for S often included -3 in the numerator.
In part (iii), the three cube roots were very often given correctly, but a surprising number of candidates had all three arguments wrong.
Part (iv) was also correctly answered by many candidates, although the connection with part (i)(B) was not always seen. Some confused w^* with w^{-1}.

4) **Polar coordinates**

This was the worst answered question, with an average mark of about 10.
In part (a)(i), most candidates did not even make the first step of expressing $r\sin\theta$ in terms of θ.
In part (a)(ii), there were some good attempts to sketch the curve, although few earned full marks; the most common error was to draw a cusp at $\theta = \frac{1}{2}\pi$.
In part (a)(iii), there was a lot of good work, and the area was often found correctly. Many made slips in the integration, and the overall factor of $\frac{1}{2}$ was sometimes missing.
In part (b)(i), the ellipse was often drawn correctly, but only a few candidates could answer parts (b)(ii) and (b)(iii).