OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS **2605**

Pure Mathematics 5

Monday **20 MAY 2002** Morning 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The approximate allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 3 printed pages and 1 blank page.

1 (a) The equation $x^3 + 4x^2 - 7 = 0$ has roots α, β and γ. Find a cubic equation with integer coefficients which has roots $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$. [4]

(b) You are given the polynomial $f(x) = kx^7 + mx^4 + 9x^2 - 95x + 360$, where k and m are constants.

When $f(x)$ is divided by $(x-2)$, the remainder is 62.

When $f(x)$ is divided by $(x+2)$, the remainder is –70.

(i) Find k and m, and show that $f'(2) = 37$. [7]

(ii) When $f(x)$ is divided by (x^2-4), the quotient is $g(x)$ and the remainder is $ax+b$, so that

$$f(x) = (x^2-4)g(x) + ax + b.$$

Find a and b. [4]

(iii) Find the remainder when $f(x)$ is divided by $(x-2)^2$. [5]

2 In this question k and θ are real numbers with $0 < k < 1$ and $0 < \theta < \frac{1}{2}\pi$.

(i) Express each of $e^{j\theta} + e^{-j\theta}$ and $(1 - ke^{j\theta})(1 - ke^{-j\theta})$ in trigonometric form. [4]

Infinite series C and S are defined by

$$C = k\cos\theta + k^2\cos 2\theta + k^3\cos 3\theta + \dots,$$
$$S = k\sin\theta + k^2\sin 2\theta + k^3\sin 3\theta + \dots.$$

(ii) Show that $C + jS$ is an infinite geometric series. [3]

(iii) By finding the sum of this series, show that $C = \dfrac{k\cos\theta - k^2}{1 - 2k\cos\theta + k^2}$ and find a similar expression for S. [9]

(iv) Given that $C = 0$, show that $S = \cot\theta$. [4]

3 (a) (i) Show that the only stationary point on the curve $y = \cosh 2x - 3\sinh x$ is $\left(\ln 2, -\frac{1}{8}\right)$. [6]

(ii) Find $\int_0^{\ln 2} (\cosh 2x - 3\sinh x)\,dx$. [4]

(b) (i) Given that $f(x) = \arctan(1 + x)$, find $f'(x)$ and $f''(x)$. [3]

(ii) Find the Maclaurin series for $\arctan(1 + x)$, as far as the term in x^2. [3]

(iii) Use this Maclaurin series to find an approximate value of $\int_0^{0.4} \arctan(1 + x^2)\,dx$, giving your answer to 3 decimal places. [4]

4 (a) $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points on the parabola $y^2 = 4ax$, with $x_2 > x_1$. The focus of the parabola is $S(a, 0)$.

Prove that $SQ - SP = x_2 - x_1$. [4]

(b) A curve C has polar equation $r = k\cos 2\theta$, for $0 \leqslant \theta \leqslant 2\pi$, where k is a positive constant. The points A and B on C correspond to $\theta = 0$ and $\theta = \frac{1}{6}\pi$ respectively.

(i) Sketch the curve C, using a continuous line for sections where $r > 0$ and a broken line for sections where $r < 0$.

Mark the points A and B on your sketch. [4]

(ii) Find the coordinates of B, in both polar and cartesian forms. [4]

(iii) Show that the area of the region enclosed by the line AB and the arc of the curve C for which $0 \leqslant \theta \leqslant \frac{1}{6}\pi$ is

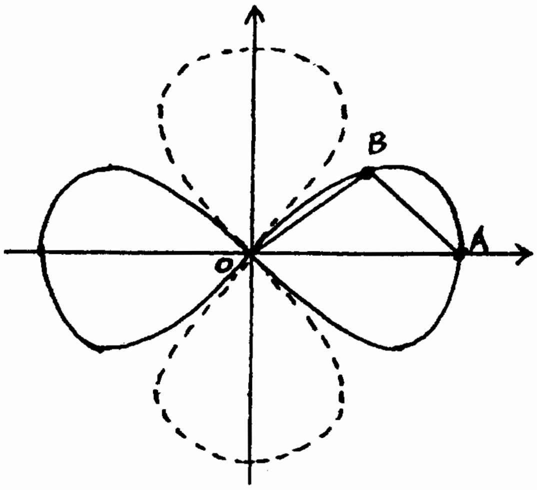
$$\frac{1}{96}\left(4\pi + 3\sqrt{3} - 12\right)k^2.$$ [8]

Mark Scheme

1 (a)	Let $y=\frac{1}{x}$, $x=\frac{1}{y}$ $\frac{1}{y^3}+\frac{4}{y^2}-7=0$ $7y^3-4y-1=0$	B2 M1 A1 4	Give B1 for $\frac{1}{y^3}$ or $\frac{4}{y^2}$ Forming a cubic equation
	OR $\sum\frac{1}{\alpha}=\frac{\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha\beta\gamma}=\frac{0}{7}=0$ $\sum\frac{1}{\beta}\frac{1}{\gamma}=\frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}=\frac{-4}{7}$ $\frac{1}{\alpha}\frac{1}{\beta}\frac{1}{\gamma}=\frac{1}{7}$ B2 Equation is $y^3-\frac{4}{7}y-\frac{1}{7}=0$ M1 $7y^3-4y-1=0$ A1		Give B1 for two correct
(b)(i)	$f(2)=62 \Rightarrow 128k+16m=-144$ $f(-2)=-70 \Rightarrow -128k+16m=-656$ Solving, $k=2,\ m=-25$ $f'(x)=14x^6-100x^3+18x-95$ $f'(2)=896-800+36-95=37$	M1 A1A1 M1 A1 M1 A1 (ag) 7	Considering $f(2)$ or $f(-2)$ Two equations Differentiating (may have *k* and *m*)
(ii)	Putting $x=2$, $62=2a+b$ Putting $x=-2$, $-70=-2a+b$ Solving, $a=33,\ b=-4$	M1 A1 M1 A1 4	Substituting $x=2$ or $x=-2$ Both equations correct
	OR By long division, M1 quotient is $2x^5+8x^3-25x^2+32x-91$ A1 M1 remainder is $33x-4$ A1		At least x^5, x^4, x^3 terms in quotient Obtaining a linear remainder
(iii)	Let $f(x)=(x-2)^2h(x)+px+q$ Putting $x=2$, $62=2p+q$ $f'(x)=(x-2)^2h'(x)+2(x-2)h(x)+p$ Putting $x=2$, $37=p$ Hence $q=-12$, and remainder is $37x-12$	B1 B1 M1 A1 B1 5	 Use of product rule *Must be correctly obtained* *Must be given as* $37x-12$ *SR* $37x-12$ with no working earns B3
	OR By long division, M1 quotient is $2x^5+8x^4+24x^3+39x^2+60x+93$ A2 M1 remainder is $37x-12$ B1		At least x^5, x^4, x^3 terms in quotient Give A1 ft for first three terms $(kx^5+4kx^4+12kx^3)$ Obtaining a linear remainder

2 (i)	$e^{j\theta} + e^{-j\theta} = (\cos\theta + j\sin\theta) + (\cos\theta - j\sin\theta)$ $= 2\cos\theta$ $(1 - ke^{j\theta})(1 - ke^{-j\theta}) = 1 - k(e^{j\theta} + e^{-j\theta}) + k^2 e^{j\theta} e^{-j\theta}$ $= 1 - 2k\cos\theta + k^2$	M1 A1 M1 A1 **4**	One term correct *May be earned later* Or any (unsimplified) trig form
(ii)	$C + jS = ke^{j\theta} + k^2 e^{2j\theta} + k^3 e^{3j\theta} + \dots$ a GP with $r = ke^{j\theta}$	M1 A1 A1 **3**	$\cos 2\theta + j\sin 2\theta = e^{2j\theta}$ *or* $(\cos\theta + j\sin\theta)^2$ *etc* *or* $ke^{j\theta} + k^2(e^{j\theta})^2 + k^3(e^{j\theta})^3 + \dots$ *SR* B2 for $k(\cos\theta + j\sin\theta) + k^2(\cos 2\theta + j\sin 2\theta) + \dots$ is a GP with $r = k(\cos\theta + j\sin\theta)$
(iii)	$C + jS = \dfrac{ke^{j\theta}}{1 - ke^{j\theta}}$ $= \dfrac{ke^{j\theta}(1 - ke^{-j\theta})}{(1 - ke^{j\theta})(1 - ke^{-j\theta})}$ $= \dfrac{ke^{j\theta} - k^2}{1 - 2k\cos\theta + k^2}$ Equating real parts, $C = \dfrac{k\cos\theta - k^2}{1 - 2k\cos\theta + k^2}$ Equating imaginary parts, $S = \dfrac{k\sin\theta}{1 - 2k\cos\theta + k^2}$	M1A1 M2A1 A1 M1 A1 (ag) A1 **9**	Give M1A0 for sum of n terms Multiplying num and denom by complex conjugate of denom Equating real or imaginary parts
(iv)	If $C = 0$, $k = \cos\theta$ Then $S = \dfrac{\cos\theta\sin\theta}{1 - 2\cos^2\theta + \cos^2\theta} = \dfrac{\cos\theta\sin\theta}{1 - \cos^2\theta}$ $= \dfrac{\cos\theta\sin\theta}{\sin^2\theta} = \cot\theta$	B1 M1A1 A1 (ag) **4**	Obtaining S in terms of θ

3 (a)(i)	$\frac{dy}{dx} = 2\sinh 2x - 3\cosh x$	M1A1	M1 for one term correct
	$= 4\sinh x\cosh x - 3\cosh x$ $= \cosh x(4\sinh x - 3)$	M1	Using $\sinh 2x = 2\sinh x\cosh x$
	Since $\cosh x \neq 0$, $\frac{dy}{dx} = 0$ only when $\sinh x = \frac{3}{4}$	A1	Requires reference to $\cosh x \neq 0$
	$x = \ln\left(\frac{3}{4} + \sqrt{(\frac{3}{4})^2 + 1}\right) = \ln 2$	A1 (ag)	*Working essential*
	OR $2e^{4x} - 3e^{3x} - 3e^x - 2 = 0$ $(e^x - 2)(2e^{3x} + e^{2x} + 2e^x + 1) = 0$ B1 $e^x = 2$ so $x = \ln 2$ B1 $2e^{3x} + e^{2x} + 2e^x + 1 \neq 0$ B1		 *Dependent on first B1* *Dependent on first B1*
	$y = \cosh(2\ln 2) - 3\sinh(\ln 2)$ $= \frac{1}{2}\left(4 + \frac{1}{4}\right) - 3\left(\frac{3}{4}\right) = -\frac{1}{8}$	B1 (ag) **6**	*Working essential*
(ii)	$\int_0^{\ln 2}(\cosh 2x - 3\sinh x)\,dx = \left[\frac{1}{2}\sinh 2x - 3\cosh x\right]_0^{\ln 2}$	M1A1	Limits not required
	$= \frac{1}{2}\sinh(2\ln 2) - 3\cosh(\ln 2) + 3$ $= \frac{1}{4}\left(4 - \frac{1}{4}\right) - \frac{3}{2}\left(2 + \frac{1}{2}\right) + 3$	M1	Substituting both limits *Dependent on previous M1*
	$= \frac{3}{16}$	A1 **4**	*SR* Give B2 for correct answer if scheme would give fewer marks
(b)(i)	$f'(x) = \frac{1}{1 + (1 + x)^2}$	B1	
	$f''(x) = \frac{-2(1 + x)}{\{1 + (1 + x)^2\}^2}$	M1 A1 **3**	Use of chain (or quotient) rule
(ii)	$f(0) = \frac{1}{4}\pi$, $f'(0) = \frac{1}{2}$, $f''(0) = -\frac{1}{2}$	M1	Evaluating (at least two) when $x = 0$
	Maclaurin series is $\arctan(1 + x) = \frac{1}{4}\pi + \frac{1}{2}x - \frac{1}{4}x^2 + \dots$	A2 ft **3**	Give A1 ft for two terms correct
(iii)	$\arctan(1 + x^2) = \frac{1}{4}\pi + \frac{1}{2}x^2 - \frac{1}{4}x^4 + \dots$	M1	
	$\int_0^{0.4}\arctan(1 + x^2)\,dx = \left[\frac{1}{4}\pi x + \frac{1}{6}x^3 - \frac{1}{20}x^5 + \dots\right]_0^{0.4}$	A1 ft	Integrating series
	OR Substituting $u = x^2$ gives $\int \frac{\arctan(1 + u)}{2\sqrt{u}}\,du = \int\left(\frac{1}{8}\pi u^{-\frac{1}{2}} + \frac{1}{4}u^{\frac{1}{2}} - \frac{1}{8}u^{\frac{3}{2}}\right)du$ M1 $= \left[\frac{1}{4}\pi u^{\frac{1}{2}} + \frac{1}{6}u^{\frac{3}{2}} - \frac{1}{20}u^{\frac{5}{2}}\right]_0^{0.16}$ A1 ft		
	$= 0.31416 + 0.01067 - 0.00051 + \dots$ ≈ 0.324 (3 dp)	M1 A1 **4**	*Dependent on previous M1* *SR* B0 for 0.324 with no working

4 (a)	SP = PM where M is $(-a,\ y_1)$ $= a + x_1$ SQ $= a + x_2$ SQ − SP $= (a + x_2) - (a + x_1)$ $= x_2 - x_1$	M1 A1 M1 A1 (ag) **4**	*Or* $\sqrt{(x_1 - a)^2 + 4ax_1}$ *Dependent on previous M1*
(b)(i)		B1 B1 B1 B1 **4**	For one loop For two further loops Fully correct with continuous and broken lines Points A and B
(ii)	$r = k\cos\frac{1}{3}\pi = \frac{1}{2}k$; polar coords are $\left(\frac{1}{2}k,\ \frac{1}{6}\pi\right)$ Cartesian coordinates are $x = r\cos\theta = \frac{1}{2}k\cos\frac{1}{6}\pi = \frac{1}{4}\sqrt{3}k$ $y = r\sin\theta = \frac{1}{2}k\sin\frac{1}{6}\pi = \frac{1}{4}k$	B1 M1 A1 A1 **4**	For $r = \frac{1}{2}k$ Use of $x = r\cos\theta$ or $y = r\sin\theta$ *SR* If k omitted, then give A1 for $r = \frac{1}{2},\ x = \frac{1}{4}\sqrt{3},\ y = \frac{1}{4}$
(iii)	Area of sector OAB is $\int_0^{\frac{1}{6}\pi} \frac{1}{2}(k\cos 2\theta)^2\,d\theta = \frac{1}{4}k^2\int_0^{\frac{1}{6}\pi}(1+\cos 4\theta)\,d\theta$ $= \frac{1}{4}k^2\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\frac{1}{6}\pi}$ $= \frac{1}{4}k^2\left(\frac{1}{6}\pi + \frac{1}{8}\sqrt{3}\right)$ Area of triangle OAB is $\frac{1}{2}k(\frac{1}{4}k) = \frac{1}{8}k^2$ Required area is (sector OAB) − (triangle OAB) $= k^2\left(\frac{1}{24}\pi + \frac{1}{32}\sqrt{3}\right) - \frac{1}{8}k^2$ $= \frac{1}{96}(4\pi + 3\sqrt{3} - 12)k^2$	M1 A1 A1A1 A1 B1 ft M1 A1 (ag) **8**	Integral of $\cos^2 2\theta$ Correct integral expression (limits required) For $\int\cos^2 2\theta\,d\theta = \frac{1}{2}(\theta + \frac{1}{4}\sin 4\theta)$ ft $\frac{1}{2}k \times (y\text{ - value})$ *or* $\frac{1}{4}k \times (r\text{ - value})$

Examiner’s Report

2605 Pure Mathematics 5

General Comments

This paper was found to be quite straightforward, compared with recent past papers. There were many excellent scripts, with a good number scoring full marks. The proportion of candidates scoring less than 20 marks (out of 60) was far smaller than usual. Q.4 was by far the least popular question.

Comments on Individual Questions

Q.1 This question was very well answered, and a good proportion of candidates scored full marks. In part (a) most used substitution, but a large number preferred to use sums of products or roots, and there were some sign errors. In part (b) the techniques were well known and often carried out accurately. A few tried to answer parts (ii) and (iii) using long division, but this was rarely free from errors.

(a) $7y^3 - 4y - 1 = 0$; (b)(i) $k = 2, m = -25$, (ii) $a = 33, b = -4$, (iii) $37x - 12$.

Q.2 This question was also answered well, with a fair proportion scoring full marks. Parts (i) and (ii) were usually answered correctly. In part (iii) a fair number of candidates worked with the sum to n terms instead of the sum to infinity, and some did not obtain an expression with a real denominator before taking real and imaginary parts. Part (iv) was very often answered correctly; when S was wrong it was still possible to earn 3 marks out of 4 by expressing their S in terms of θ.

(i) $2\cos\theta$, $1 - 2k\cos\theta + k^2$, (iii) $S = \dfrac{k\sin\theta}{1 - 2k\cos\theta + k^2}$.

Q.3 This was not as well answered as Q.1 and Q.2. Not many candidates scored full marks in part (a)(i); as the answer is given, the marks are awarded for the working, which therefore needs to be sufficiently clear and complete. The main failings were: not using $\sinh 2x = 2\sinh x\cosh x$ (it is possible to solve the equation by converting to exponential form, but this proved to be much more difficult); cancelling the factor $\cosh x$ without any explanation; just verifying by substitution that $x = \ln 2$ gives zero gradient (this does not prove that it is the *only* stationary point); stating $y = -\frac{1}{8}$ without any working. By contrast, part (a)(ii) was usually answered correctly, although many candidates omitted to substitute the lower limit $x = 0$ into the integral. The derivatives and Maclaurin series in parts (b)(i) and (ii) were usually found correctly, but in part (b)(iii) very many did not realise that they needed to replace x in the series by x^2. They either integrated the series for $\arctan(1 + x)$ or started again and attempted to find the Maclaurin series for $\arctan(1 + x^2)$ directly.

(a)(ii) $\frac{3}{16}$; (b)(i) $f'(x) = \dfrac{1}{1 + (1 + x)^2}$, $f''(x) = \dfrac{-2(1 + x)}{\{1 + (1 + x)^2\}^2}$, (ii) $\frac{1}{4}\pi + \frac{1}{2}x - \frac{1}{4}x^2 + \ldots$, (iii) 0.324.

Q.4 This question was attempted by about one quarter of the candidates. It was the worst answered question, with most attempts scoring less than half marks. Part (a) was poorly answered. Often the directrix was drawn as the tangent at the vertex, followed by SP $= x_1$ and SQ $= x_2$. In part (b)(i) the curve was usually substantially correct, although the continuous and broken lines were often not in the right places. In part (b)(ii) the coordinates were usually found correctly. In part (b)(iii) not many scored full marks; there were difficulties in integrating $\cos^2 2\theta$, and the triangle was often forgotten.

(b)(ii) $\left(\frac{1}{2}k, \frac{1}{6}\pi\right)$ $\left(\frac{1}{4}\sqrt{3}k, \frac{1}{4}k\right)$.