OCR RECOGNISING ACHIEVEMENT

Oxford, Cambridge and RSA Examinations

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2602/1

Pure Mathematics 2

Thursday **10 JANUARY 2002** Morning 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer **all** questions.
- You are permitted to use only a scientific calculator in this paper.

INFORMATION FOR CANDIDATES

- The approximate allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 5 printed pages and 3 blank pages.

1 **(a)** Express $\dfrac{\sqrt{2}}{\sqrt{2}+1}$ in the form $a+b\sqrt{2}$, where a and b are integers. [3]

(b) Solve the equation $x^{\frac{3}{2}} = 20$, giving your answer to 3 significant figures. [2]

(c) Differentiate $x\ln(1+x^2)$ with respect to x. [4]

(d) Fig. 1 shows a sketch of the graph of a function f(x).

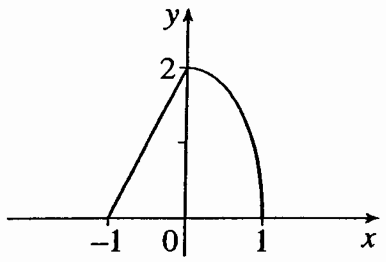


Fig. 1

Sketch the graph of f(x + 1). [2]

(e) Given that $y=\sqrt{x+1}$,

(A) find x in terms of y,

(B) find $\dfrac{dx}{dy}$ in terms of y,

(C) find $\dfrac{dy}{dx}$ in terms of x. [4]

2 Fig. 2 shows a sketch of the graph of $y = \dfrac{x}{(2x-1)^2}$.

P is a stationary point on the curve. The line $x = a$ is an asymptote to the curve.

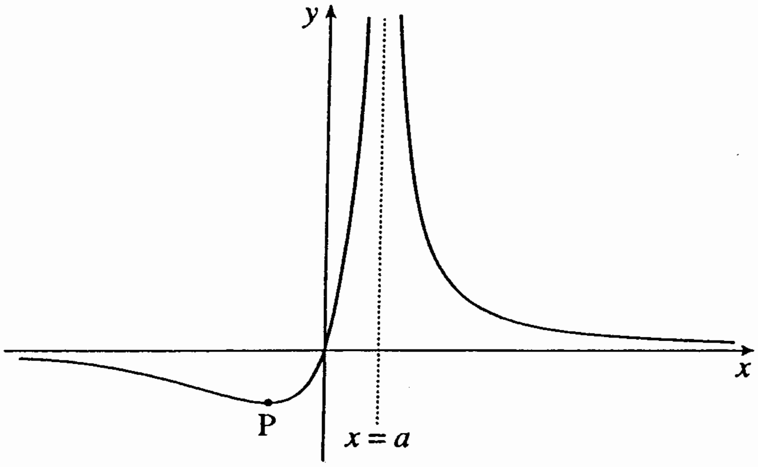


Fig. 2

(i) Write down the value of a. **[1]**

(ii) Show that $\dfrac{dy}{dx} = -\dfrac{2x+1}{(2x-1)^3}$.

Hence find the coordinates of P. **[7]**

The area of the region enclosed by the graph of $y = \dfrac{x}{(2x-1)^2}$, the x-axis, and the lines $x = 1$ and $x = 2$ is denoted by A.

(iii) Using a suitable substitution, show that $A = \displaystyle\int_1^3 \frac{u+1}{4u^2}\,du$.

Deduce that $A = \dfrac{3\ln 3 + 2}{12}$. **[7]**

3 The temperature θ °C of a cup of tea t minutes after pouring is modelled by the equation

$$\theta = 25 + 70\,e^{-0.1t}.$$

(i) What is the initial temperature of the tea? [1]

(ii) Find $\frac{d\theta}{dt}$. Hence calculate the initial rate of cooling of the tea. [3]

(iii) After the tea has been cooling for 3 minutes, milk is added. This causes an instantaneous drop in temperature of 5 °C. Show that the temperature of the tea is now 72 °C, to the nearest degree. [2]

On another occasion, the milk is added immediately after pouring the tea. This causes the initial temperature to drop by 5 °C. Subsequently, the temperature is modelled by the equation

$$\theta = 25 + k\,e^{-0.1t}.$$

The initial temperature of the tea is the same as in part **(i)**.

(iv) Show that $k = 65$, and calculate the initial rate of cooling (after the milk is added). [3]

(v) Calculate how long it takes for the temperature to drop to 72 °C. [4]

(vi) (*A*) Explain the significance of the constant 25 in the equations for θ. [1]

(*B*) According to these models, the tea takes longer to cool to 72 °C when the milk is added immediately. By comparing your expressions for $\frac{d\theta}{dt}$ in parts **(ii)** and **(iv)**, suggest a reason for this. [1]

4 **(a)** A sequence is defined by

$$u_{r+1} = u_r - 3, \qquad u_1 = 102.$$

(i) Find u_2, u_3 and u_{100}. [2]

(ii) Find an expression for $\sum_{r=1}^{n} u_r$ in terms of n, simplifying this as far as possible.

Find the value of n for which $\sum_{r=1}^{n} u_r = 0$. [4]

(b) A ball crosses backwards and forwards between two walls A and B, starting at A (see Fig. 4). Each time it rebounds from a wall, its speed is reduced. The times for successive crossings form a geometric progression with first term 2 and common ratio 1.2.

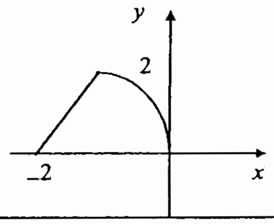
Fig. 4

(i) Show that the ball reaches B for the second time after a total time of 7.28 seconds. [1]

(ii) Find the total time taken after the ball has crossed for the 20th time, giving your answer to the nearest second. [3]

(iii) Find how many crossings the ball makes in the first fifteen *minutes*. [5]

Mark Scheme

1(a) $\frac{\sqrt{2}}{\sqrt{2}+1}=\frac{\sqrt{2}(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)}$ $=\frac{2-\sqrt{2}}{2-1}$ $=2-\sqrt{2}$	M1 A1 A1 cao [3]	$\times$ top and bottom by $\sqrt{2}-1$ or $1-\sqrt{2}$ or $\frac{\sqrt{2}-2}{1-2}$ allow $2-1\sqrt{2}$ Unsupported correct answers SCB2
(b) $x^{3/2}=20$ $\Rightarrow x=20^{2/3}$ $=7.37$ (3 s.f.)	M1 A1 cao [2]	or $1.5\ln x=\ln 20$ or trial and improvement or $x^3=400$ Allow unsupported correct answers
(c) $\frac{dy}{dx}=x.\frac{2x}{1+x^2}+\ln(1+x^2)$ $=\frac{2x^2}{1+x^2}+\ln(1+x^2)$	M1 B1 A1 A1 [4]	product rule attempted with $u=x$ and $v=\ln(1+x^2)$ derivative of $\ln(1+x^2)=\frac{2x}{1+x^2}$ soi $\frac{2x^2}{1+x^2}$ $+\ln(1+x^2)$
(d) 	M1 A1 [2]	Translation horizontally 1 unit to left
(e) $y=\sqrt{x+1}$ $\Rightarrow y^2=x+1, \Rightarrow x=y^2-1$ $\Rightarrow\frac{dx}{dy}=2y$ $\Rightarrow\frac{dy}{dx}=\frac{1}{2y}$ $=\frac{1}{2\sqrt{x+1}}$	B1 B1ft M1 A1 cao [4]	 on reasonable x s or for use of chain rule or $\frac{1}{2}(x+1)^{-1/2}$

2(i) $a = 1/2$	B1 [1]	
(ii) $\frac{dy}{dx} = \frac{(2x-1)^2.1 - x.2(2x-1).2}{(2x-1)^4}$ $= \frac{(2x-1)(2x-1-4x)}{(2x-1)^4}$ $= -\frac{(2x+1)}{(2x-1)^3}$ *	B1 M1 M1 E1 (4)	derivative of $(2x-1)^2$ is $2(2x-1).2$ numerator consistent with their derivatives denominator correct (or $[(2x-1)^2]^2$) www (not $\frac{-2x+1}{(2x-1)^3}$)
Or $\frac{dy}{dx} = x(-2)(2x-1)^{-3}.2 + (2x-1)^{-2}$ $= (2x-1)^{-3}(-4x+2x-1)$ $= (2x-1)^{-3}(-1-2x)$ $= -\frac{1+2x}{(2x-1)^3}$	B1 M1 M1 E1 (4)	$-4(2x-1)^{-3}$ Product rule consistent with their derivatives Factor of $(2x-1)^{-3}$ or common denominator
$\frac{dy}{dx} = 0$ when $2x + 1 = 0$ $\Rightarrow x = -1/2$, $y = -1/8$	M1 A1 cao A1 cao (3) [7]	setting *numerator* to 0
(iii) $A = \int_1^2 \frac{x}{(2x-1)^2} dx$ Let $u = 2x - 1$, $du = 2dx$ when $x = 1$, $u = 1$ when $x = 2$, $u = 3$ so $A = \int_1^3 \frac{\frac{1}{2}(u+1)}{u^2} \frac{1}{2} du$ $= \int_1^3 \frac{(u+1)}{4u^2} du$ *	M1 M1 E1 E1 (4)	Correct integral (soi) -condone no dx $u = 2x - 1$ changing limits – must show some working transforming integral (www)
$= \frac{1}{4}\int_1^3 (\frac{1}{u} + \frac{1}{u^2}) du$ $= \frac{1}{4}\left[\ln u - \frac{1}{u}\right]_1^3$ $= \frac{1}{4}(\ln 3 - \frac{1}{3} - \ln 1 + 1)$ $= \frac{1}{4}(\ln 3 + \frac{2}{3})$ $= \frac{3\ln 3 + 2}{12}$ *	M1 A1 E1 (3) [7]	splitting fraction correctly integrated www

3 (i) $\theta = 25 + 70 = 95°C$	B1 [1]	
(ii) $\frac{d\theta}{dt} = -7e^{-0.1t}$ $= -7$ when $t = 0$	M1 A1 B1ft [3]	chain rule attempted on $e^{-0.1t}$ $-7e^{-0.1t}$ -7 or 7 allow ft on correct evaluations of their derivative at $t = 0$, unless zero.
(iii) When $t = 3$, $\theta = 25 + 70\ e^{-0.3}$ $= 76.86$ $76.86 - 5 = 71.86$ $= 72°C$ to nearest degree	M1 E1 [2]	Substituting $t = 3$
(iv) Initially $\theta = 90$ $\Rightarrow 90 = 25 + k$ $\Rightarrow k = 65$ * $\frac{d\theta}{dt} = -6.5\ e^{-0.1t}$ $= -6.5$ when $t = 0$	 B1 M1 A1cao [3]	 -6.5 or 6.5
(v) $25 + 65\ e^{-0.1t} = 72$ $\Rightarrow e^{-0.1t} = 47/65$ $\Rightarrow -0.1t = \ln(47/65)$ $\Rightarrow t = -10\ \ln(47/65)$ $= 3.24$ mins	M1 A1 M1 A1 cao [4]	equating to 72 taking logs or trial and improvement 3.2(4…) or 3 mins 12 s – 3 mins 15s (mark the decimal) If trial and improvement used on equation without rearrangement, M1A2
(vi) (*A*) As $t \to \infty$, $\theta \to 25$ so 25 is the room temperature (*B*) Rate of cooling is less when milk is added first.	 B1 B1 [2]	 or long term or minimum value of θ or correct comparison of gradients

4(a) (i) $u_2 = 102 - 3 = 99$ $u_3 = 99 - 3 = 96$ $u_{100} = 102 + 99(-3)$ $= -195$	B1 B1 [2]	99, 96
(ii) $\sum_{r=1}^{n} u_r = \frac{n}{2}(204 + [n-1](-3))$ $= \frac{n}{2}(207 - 3n)$ $\sum_{r=1}^{n} u_r = 0$ when $207 - 3n = 0$ $\Rightarrow n = 69$	M1 A1 M1 ft A1cao [4]	sum of AP with $a = 102$, $d = -3$ [condone $d = 3$] setting their sum formula to zero (not term)
(b)(i) $2 + 2 \times 1.2 + 2 \times 1.2^2$ (= 7.28*)	B1 [1]	or $\frac{2(1-1.2^3)}{(1-1.2)}$ or $\frac{2(1.2^3-1)}{(1.2-1)}$ (= 7.28*)
(ii) $2 + 2 \times 1.2 + \ldots + 2 \times 1.2^{19}$ $= \frac{2(1.2^{20}-1)}{(1.2-1)}$ = 373 secs	M1 A1 A1 cao [3]	Use of sum of GP formula [condone 1.2^{19} or 1.2^{21}) or $\frac{2(1-1.2^{20})}{(1-1.2)}$
(iii) $\frac{2(1.2^n-1)}{(1.2-1)} = 10(1.2^n - 1) = 900$	M1	Equating correct expression to 900
$\Rightarrow 1.2^n = 91$ $\Rightarrow n \ln 1.2 = \ln 91$ $\Rightarrow n = \frac{\ln 91}{\ln 1.2} = 24.74$ So $n = 24$	A1 M1 ft A1 ft A1 cao	$1.2^n = 91$ taking logs with n down 24.74 soi
or $n = 24 \Rightarrow 784.96\ldots$ $n = 25 \Rightarrow 943.96\ldots$ so 24 crossings	M1 ft A1 ft A1 ft A1 cao	Trial and improvement (at least 2 values) on GP sum formula (not term) value immediately below 900 value immediately above 900 SCB1 for $n = 24$ unsupported
or $1.2^n = 91$ $n = 24 \Rightarrow 79.496..$ $n = 25 \Rightarrow 95.396..$ so 24 crossings	A1 ft M1 ft A1ft A1 cao [5]	 Trial and improvement (at least 2 values) values of n immediately below and above their 91 SCB1 for $n = 24$ unsupported

Examiner’s Report

Pure Mathematics 2 (2602)

General Comments

A very disparate standard of work was seen, with many excellent scores of over 50 and a significant 'tail' of candidates who scored less than 10 marks and demonstrated very little knowledge of the syllabus. All four questions proved to be accessible to candidates, and there was little difference in the average marks scored per question. The new-style question 1 scored slightly higher than the other questions, and succeeded in giving candidates straightforward short tests of basic syllabus content. This question also allows the testing of topics, such as surds, which do not readily fit into a 15-mark long question. In the answers to question 4, there were signs of some candidates running out of time, and the marks for this question reflected this, albeit not significantly. In general, though, most candidates had ample time to complete the paper.

Fragile algebra, as usual, proved to be the main weakness of the work seen. Students need to be able to manipulate algebraic fractions to apply the calculus, and many do not have confidence here, as can be seen from questions 2(ii) and 2(iii). In question 4, solving $\frac{1}{2}n(207 - 3n) = 0$ by expanding and then using a quadratic formula shows a worrying level of algebraic naivety!

Results, when given, are intended to be helpful and to ensure that subsequent parts are accessible to candidates. However, it is important that able candidates are careful to give sufficient working when deriving these, for example showing the change of limits in 2(iii).

Comments on Individual Questions

Question 1 (Various)

(*a*) Although a standard piece of surd manipulation, this question was very poorly answered, perhaps because it has not been tested frequently. Candidates usually gained all three marks or none, depending on whether they were familiar with the method of multiplying top and bottom by $\sqrt{2} - 1$.

$[a = 2, b = -1]$

(*b*) This was well answered, with the occasional loss of a mark for failing to give the answer to three significant figures.

$[x = 7.37]$

(*c*) Weaker candidates often misinterpreted this product rule as 'x ln' multiplied by '$1+x^2$'! Another error was to omit the derivative of x^2 when differentiating $\ln(1 + x^2)$. However, there were many correct solutions.

$$\left[\frac{dy}{dx} = \frac{2x^2}{1+x^2} + \ln(1+x^2)\right].$$

(*d*) This question was very well answered.

(*e*) Again, there were many correct solutions. Most candidates differentiated $\sqrt{x+1}$ directly rather than using the reciprocal of $\frac{dx}{dy}$.

Question 2 (Calculus)

(i) This was intended to be an easy starter for 1 mark, which was usually, although not universally, gained.

(ii) The quotient rule was generally used correctly, with occasional use of the product rule. However, many candidates failed to derive the final form of the derivative convincingly, clearly extracting the negative sign. The answer $\frac{dy}{dx} = \frac{-2x+1}{(2x-1)^3}$ was not accepted. Most set the numerator to zero and found the correct turning point, although there was some evidence of 'fiddling', e.g. $\frac{-2x+1}{(2x-1)^3} = 0 \Rightarrow -2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$. Setting the denominator to zero scored M0.

(iii) This is the easiest type of substitution, but was often poorly answered. Candidates really do need to show the 'dx' in the integrand clearly substituted by '½ du' – many students just omit the dx and the du altogether. We also needed to see clear evidence of transforming the limits (e.g. when $x = 1$, $u = 1$, etc). As for the integration itself, this was very poorly done – very few candidates split the fraction before integrating. This lack of facility with algebraic fractions remains a problem to be tackled.

[Turning point is $(-\frac{1}{2}, -\frac{1}{8})$]

Question 3 (Exponential functions)

(i) This was an easy starter mark, which required candidates to know that $e^0 = 1$.

(ii) The derivative of $70e^{-0.1t}$ was frequently incorrect, with $-7t\,e^{-0.1t}$ a common error. Substituting $t = 0$ also required $e^0 = 1$.

(iii) This was well answered.

(iv) Finding $k = 65$ needed to be from $90 = 25 + k$ – some stated that $95 = 25 + k$, then took off 5 degrees. The initial rate suffered from similar mistakes to the previous part.

(v) There were plenty of correct answers to this part. Marks were usually lost for taking logs of each side incorrectly.

(vi) Room temperature was well interpreted. In part (*B*), candidates needed to refer specifically to the different rates of cooling, or gradients of the cooling curves.

[95°, –7° per min, –6.5° per min, 3.24 mins]

Question 4 (Sequences and series)

(*a*) (i) This was usually correct, although $u_{100} = -198$ was an occasional error.

(ii) Using the term formula here instead of the sum gained no marks. The arithmetic series formula wa usually known, with occasional slips in dealing with the –3 in the substitution. Many candidates mad rather heavy weather of the quadratic, for example expanding the bracket and then using the quadrati formula!

(*b*) (i) This was a well answered, easy mark.

(ii) The geometric series formula was well known and evaluated correctly.

(iii) Most candidates attempted to solve this by taking logarithms, often successfully, although a number le the answer as 24.74 rather than stating 24 *complete* crossings. For trial and improvement, they needed t evaluate $n = 24$ and $n = 25$.

[99, 96, –195 ; $n = 69$; 373 seconds, 24 crossings]