OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2610/1

Differential Equations (Mechanics 4)

Friday **14 JANUARY 2005** Morning 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- Take g = 9.8 m s^{-2} unless otherwise instructed.
- The total number of marks for this paper is 60.

This question paper consists of 4 printed pages.

1 The differential equation $\frac{dy}{dx} + 2xy = e^{-(x-2)^2}$ is to be solved for $x > 1$.

(i) Find the general solution for y in terms of x. [7]

(ii) Find the particular solution subject to the condition $y = 0$ when $x = 1$. Show that $y > 0$ for $x > 1$. Sketch the solution curve for $x \geqslant 1$. [6]

(iii) Given instead the condition that y is at its maximum value when $x = 2$, use the original differential equation to show that the maximum value of y is $\frac{1}{4}$.

Hence, or otherwise, find the particular solution. Sketch the solution curve. [7]

2 The wave function, y, of a radioactive particle satisfies the differential equations

$$\frac{d^2y}{dx^2} + 9y = 0 \qquad \text{for } x < 0,$$

$$\frac{d^2y}{dx^2} - (k-9)y = 0 \qquad \text{for } x > 0.$$

(i) In the case $k = 25$, find the general solution of each equation. [6]

The solutions found in part (i) satisfy the following conditions:

(1) each solution gives $y = y_0$ when $x = 0$,

(2) both solutions give the same value for $\frac{dy}{dx}$ when $x = 0$,

(3) y is bounded for all values of x.

(ii) Use these conditions to determine the arbitrary constants in the general solutions in part (i) in terms of y_0. [8]

(iii) Sketch the graph of y against x for the case $y_0 > 0$. [3]

(iv) In the case $k = 5$, find the general solution for $x > 0$.

State with reasons whether or not the given conditions are enough to determine the arbitrary constants in this case. [3]

3 Water is draining from a small hole near the base of a large barrel. At time t, the speed of the flow of water is $v\ \mathrm{m\,s^{-1}}$, the volume of water in the barrel is $V\ \mathrm{m^3}$ and the height of water above the hole is x m. The hole has a cross-section of area $0.0004\ \mathrm{m^2}$.

Torricelli's law states that $v = \sqrt{2gx}$.

Initially the height of water above the hole is 2 m.

(i) Show that $\frac{dV}{dx}\frac{dx}{dt} = -0.0004\sqrt{2gx}$. [3]

(ii) The barrel is modelled initially by taking $V = \frac{5}{3}x$. Find x in terms of t. Calculate the time for the barrel to empty. [6]

(iii) A refined model gives $V = x + x^2 - \frac{1}{3}x^3$. Calculate the time for the barrel to empty. [6]

(iv) To take account of fluctuations in the flow, the model is refined further to give

$$(1 + 2x - x^2)\frac{dx}{dt} = -0.0004\left(\sqrt{2gx} + 0.1\sin t\right).$$

Euler's method is used to estimate x. The algorithm is given by

$$x_{r+1} = x_r + h\dot{x}_r \qquad t_{r+1} = t_r + h,$$

where h is the step length. The following results are calculated.

t	x	$\dot{x}$
0	2	−0.00250
0.1	1.99975	
0.2		

Verify the calculations for the first step and then calculate one more step to estimate x when $t = 0.2$. [5]

4 The simultaneous differential equations

$$\frac{dx}{dt} - 8x + 3y = 0 \qquad (1)$$

$$\frac{dy}{dt} + 2x - 7y = 0 \qquad (2)$$

are to be solved.

(i) Eliminate y from the equations to show that $\frac{d^2x}{dt^2} - 15\frac{dx}{dt} + 50x = 0$. [5]

(ii) Find the general solution for x. Use this solution and equation (1) to find the corresponding general solution for y. [6]

Now consider the following simultaneous differential equations.

$$\frac{dx}{dt} - 8x + 3y = e^{-t} \qquad (3)$$

$$\frac{dy}{dt} + 2x - 7y = e^{-t} \qquad (4)$$

(iii) Given that these equations have a solution of the form $x = ae^{-t}, y = be^{-t}$, calculate the values of a and b. [5]

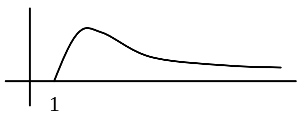
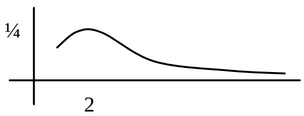
(iv) Denoting the solutions in part **(ii)** by $x = f(t)$ and $y = g(t)$, show that

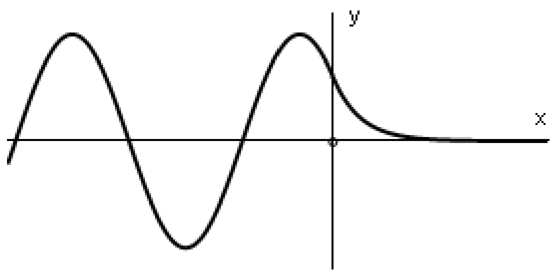
$$x = f(t) + ae^{-t}$$

$$y = g(t) + be^{-t}$$

are the general solutions of the differential equations (3) and (4). [4]

Mark Scheme

1(i)	$I = \exp\left(\int 2x\,\mathrm{d}x\right)$	M1	
	$= \mathrm{e}^{x^2}$	A1	
	$\mathrm{e}^{x^2}\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2x\,\mathrm{e}^{x^2}y = \mathrm{e}^{x^2-(x-2)^2}$	M1	multiply
	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^2}y\right) = e^{4x-4}$	F1	follow their integrating factor
	$\mathrm{e}^{x^2}y = \int \mathrm{e}^{4x-4}\,\mathrm{d}x$	M1	integrate
	$= \frac{1}{4}\mathrm{e}^{4x-4} + A$	A1	
	$y = \mathrm{e}^{-x^2}\left(\frac{1}{4}\mathrm{e}^{4x-4} + A\right)$	F1	divide by their I (must divide constant also)
			7
(ii)	$0 = \mathrm{e}^{-1}\left(\frac{1}{4} + A\right) \Rightarrow A = -\frac{1}{4}$	M1	condition on y
	$y = \frac{1}{4}\mathrm{e}^{-x^2}\left(\mathrm{e}^{4x-4} - 1\right)$	A1	cao
	$x > 1 \Rightarrow 4x - 4 > 0 \Rightarrow \mathrm{e}^{4x-4} > 1$	M1	attempt inequality for y
	and $\frac{1}{4}\mathrm{e}^{-x^2} > 0$ so $y > 0$	E1	fully justified
		B1	through (1,0) and $y > 0$ for $x > 1$
		B1	asymptotic to $y = 0$
			6
(iii)	maximum $\Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$	M1	
	$\Rightarrow 2xy = \mathrm{e}^{-(x-2)^2}$	M1	must use DE
	$x = 2 \Rightarrow 4y = 1 \Rightarrow y = \frac{1}{4}$	E1	
	$\frac{1}{4} = \mathrm{e}^{-4}\left(\frac{1}{4}\mathrm{e}^{4} + A\right)$	M1	substitute into GS for y
	$\Rightarrow A = 0 \Rightarrow y = \frac{1}{4}\mathrm{e}^{-(x-2)^2}$	A1	cao
		B1	general shape consistent with their solution
		B1	maximum labelled at $(2, \frac{1}{4})$
			7

2(i)	for $x < 0$, aux.eq. $\alpha^2 + 9 = 0 \Rightarrow \alpha = \pm 3j$	M1	imaginary root or recognise SHM equation	
	$y = A\cos 3x + B\sin 3x$	A1		
	for $x > 0$, $\frac{d^2y}{dx^2} - 16y = 0$	B1	may be implied	
	$\alpha^2 - 16 = 0$	M1		
	$\alpha = \pm 4$	A1		
	$y = C\,e^{-4x} + D\,e^{4x}$	F1	accept *A*,*B* again here but not in (ii)	
				6
(ii)	(1) $A = y_0$	F1		
	$C + D = y_0$	F1		
	(2) $x < 0, \frac{dy}{dx} = -3A\sin 3x + 3B\cos 3x$	M1	differentiate	
	$x > 0, \frac{dy}{dx} = -4C\,e^{-4x} + 4D\,e^{4x}$	M1	differentiate	
	$3B = -4C + 4D$	A1		
	(3) only e^{4x} is unbounded so $D = 0$	B1		
	hence $C = y_0$	B1		
	$B = -\frac{4}{3}y_0$	B1		
				8
(iii)		B1 B1 B1	curve for $x < 0$ curve for $x > 0$ continuous gradient at $x = 0$ (must have reasonable attempt at each of $x < 0$ and $x > 0$)	
				3
(iv)	$\frac{d^2y}{dx^2} + 4y = 0 \Rightarrow y = E\cos 2x + F\sin 2x$	B1		
	bounded so (3) provides no equation	M1		
	hence insufficient (3 equations, 4 unknowns)	A1		
				3

3(i)	rate of flow = area × speed $\Rightarrow \dfrac{\mathrm{d}V}{\mathrm{d}t} = -0.0004\sqrt{2gx}$	M1	accept either in words or symbols for M1 but need both for E1	
	$\Rightarrow \dfrac{\mathrm{d}V}{\mathrm{d}x}\dfrac{\mathrm{d}x}{\mathrm{d}t} = -0.0004\sqrt{2gx}$	M1 E1	use of chain rule complete argument	
				3
(ii)	$\dfrac{\mathrm{d}V}{\mathrm{d}x} = \frac{5}{3} \Rightarrow \frac{5}{3}\dfrac{\mathrm{d}x}{\mathrm{d}t} = -0.0004\sqrt{2gx}$	B1		
	$\int \frac{5}{3}x^{-1/2}\mathrm{d}x = \int -0.0004\sqrt{2g}\ \mathrm{d}t$	M1	separate	
	$\frac{10}{3}x^{1/2} = -0.0004\sqrt{2g}\,t + A$	M1	integrate	
	$t = 0, x = 2 \Rightarrow A = \frac{10}{3}\sqrt{2}$	M1	condition on x	
	$x = 2\left(1 - 0.00012\sqrt{g}\,t\right)^2$	A1	cao	
	$x = 0 \Rightarrow t \approx 2660$	A1		
				6
(iii)	$\left(1 + 2x - x^2\right)\dfrac{\mathrm{d}x}{\mathrm{d}t} = -0.0004\sqrt{2gx}$	M1	substitute for $\dfrac{\mathrm{d}V}{\mathrm{d}x}$	
	$\int (x^{-1/2} + 2x^{1/2} - x^{3/2})\mathrm{d}x = \int -0.0004\sqrt{2g}\ \mathrm{d}t$	M1	separate	
	$2x^{1/2} + \frac{4}{3}x^{3/2} - \frac{2}{5}x^{5/2} = -0.0004\sqrt{2g}\,t + B$	M1 A1	integrate	
	$t = 0, x = 2 \Rightarrow B = \frac{46}{15}\sqrt{2} \approx 4.337$	M1	condition on x	
	$x = 0 \Rightarrow t \approx 2450$	A1		
				6
(iv)	$\dot{x}(0) = -0.0004(\sqrt{4g} + 0)/1 = -0.00250$	E1	must show working	
	$x(0.1) = 2 - 0.00250 \times 0.1$	M1	use of algorithm	
	$=1.99975$	E1	must show working	
	$\dot{x}(0.1) = -0.00251$	B1		
	$x(0.2) = 1.99950$	B1		
				5

4(i)	$\frac{d^2x}{dt^2} = 8\frac{dx}{dt} - 3\frac{dy}{dt}$	M1 A1	differentiate	
	$= 8\frac{dx}{dt} - 3(-2x + 7y)$	M1	substitute for $\frac{dy}{dt}$	
	$= 8\frac{dx}{dt} + 6x - \frac{21}{3}\left(8x - \frac{dx}{dt}\right)$	M1	substitute for y	
	$\frac{d^2x}{dt^2} - 15\frac{dx}{dt} + 50x = 0$	E1		
				5
(ii)	$\alpha^2 - 15\alpha + 50 = 0$	M1	auxiliary equation	
	$\alpha = 5$ or 10	A1		
	$x = Ae^{5t} + Be^{10t}$	F1	CF for their roots	
	$y = \frac{1}{3}(8x - \dot{x})$	M1	rearrange equation (1)	
	$= \frac{1}{3}\left(8Ae^{5t} + 8Be^{10t} - \left(5Ae^{5t} + 10Be^{10t}\right)\right)$	M1	substitute for x and $\dot{x}$	
	$= Ae^{5t} - \frac{2}{3}Be^{10t}$	A1	cao	
				6
(iii)	$-ae^{-t} = 8ae^{-t} - 3be^{-t} + e^{-t}$	M1	substitute in (3)	
	$-be^{-t} = -2ae^{-t} + 7be^{-t} + e^{-t}$	M1	substitute in (4)	
	$-9a + 3b = 1, \quad 2a - 8b = 1$	M1	compare coefficients and solve	
	$a = -\frac{1}{6}$	A1		
	$b = -\frac{1}{6}$	A1		
				5
(iv)	substituting f(t), g(t) into LHS gives zero	B1	stated (or substitution)	
	substituting solutions in (iii) gives e^{-t}	B1		
	as equations linear, substituting sums gives sum $0 + e^{-t} = e^{-t}$	E1	(or verified by substitution)	
	General solutions because two arbitrary constants as expected for two first order equations.	E1		
				4

Examiner’s Report