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RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2610/1

Differential Equations (Mechanics 4)

Wednesday **23 JUNE 2004** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.
- There is an **insert** for use in Question **4** parts **(i)** and **(ii)**.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- Take g = 9.8 m s^{-2} unless otherwise instructed.
- The total number of marks for this paper is 60.

This question paper consists of 4 printed pages and an insert.

1 A partly filled cylindrical oil drum of mass 300 kg floats with its axis vertical in a lake. The oil drum experiences an upthrust of magnitude ky N from the water, where y m is the depth of the submerged part of the drum and k is a constant.

The oil drum is 1 m high and, when floating in equilibrium, 0.8 m of it is below the surface of the lake.

(i) Show that $k = 3675$. [2]

The oil drum is pushed down until its upper end is just level with the surface of the lake. The drum is then released from rest. It is subject to a resistance to motion of magnitude $300v$ N, where $v \text{ m s}^{-1}$ is the speed of the drum t seconds after it is released.

Hence the motion of the drum is described by the differential equation

$$300\frac{d^2y}{dt^2} = 2940 - 300\frac{dy}{dt} - 3675y.$$

(ii) Explain briefly the significance of each of the terms, including their signs. State the initial conditions. [5]

(iii) Illustrate by sketch graphs the difference between an over-damped system and an under-damped system. Without solving the differential equation determine the type of damping for the oil drum. [4]

(iv) Solve the differential equation to find y in terms of t. [9]

2 A chemical reaction involving compounds X, Y and Z is modelled by the equations

$$\frac{dx}{dt} = -5x, \qquad (1)$$

$$\frac{dy}{dt} = x - 2y + 4z, \qquad (2)$$

$$\frac{dz}{dt} = x + 3y - 6z, \qquad (3)$$

where x, y and z are the quantites of the compounds X, Y and Z respectively and t is the time from the start of the reaction.

Initially $x = 3$ and there is no Y and no Z present.

(i) Show that $x = 3e^{-5t}$. [2]

(ii) Eliminate z between equations (2) and (3) to show that $\frac{d^2y}{dt^2} + 8\frac{dy}{dt} = 15e^{-5t}$. [5]

(iii) Find the general solution for y in terms of t. Show that when $t = 0$, $\frac{dy}{dt} = 3$ and hence find the particular solution. [10]

(iv) Without solving for z, show that $x + 3y + 2z = 3$. Hence find z in terms of t. [3]

3 The motion of a simple pendulum is modelled neglecting air resistance, leading to the differential equation

$$\frac{d^2\theta}{dt^2} = -16 \sin \theta, \qquad (1)$$

where θ is the angle the pendulum makes with the downward vertical at time t. The pendulum is released from rest at an angle $\theta = \alpha$ where $0 < \alpha < \frac{1}{2}\pi$.

(i) Given that θ remains small throughout the motion, use the result that $\sin\theta \approx \theta$ to write down an approximate differential equation for θ. Give the solution for this equation and hence calculate the time that elapses from the release of the pendulum until it is first vertical. [4]

Another equation (the energy equation) for the pendulum is

$$\left(\frac{d\theta}{dt}\right)^2 = 32(\cos\theta - \cos\alpha). \qquad (2)$$

(ii) Deduce from this equation that $|\theta| \leqslant \alpha$ throughout the motion. For what values of θ is the pendulum instantaneously at rest? [4]

Equation (2) is to be solved numerically in the case $\alpha = \frac{1}{4}\pi$ to find the time, T_1, that elapses from the release of the pendulum until it is first vertical.

Euler's method, given by the algorithm $\theta_{r+1} = \theta_r + h\dot{\theta}_r, t_{r+1} = t_r + h$, is used.

(iii) Describe the problem that occurs if the algorithm is applied starting from $t = 0, \theta = \frac{1}{4}\pi$. [2]

(iv) Explain how this problem can be overcome by applying the algorithm from $t = 0, \theta = 0$, with $\dot{\theta}$ positive. [1]

(v) Applying the algorithm in this way, with a step length $h = 0.01$, gives $\theta = 0.7813$ when $t = 0.37$. Carry out two more steps of the algorithm. Hence state an estimate for T_1, justifying your answer. [6]

(vi) Given that $\displaystyle\int_0^{\frac{1}{4}\pi} \frac{1}{\sqrt{\cos\theta - \cos\frac{1}{4}\pi}}\,d\theta \approx 2.3012$, calculate another estimate for T_1. [3]

4 **Answer part (i) and part (ii)(*C*) of this question on the insert provided.**

The differential equation

$$\frac{dy}{dx} + ky = 10\sin 4x, \qquad x \geqslant 0,$$

is to be investigated for $k = -3$ and $k = 3$.

The insert shows the tangent field for each of the two cases.

(i) On the insert, sketch the solution curve through $(0, 1)$ for each of the two cases. Describe the difference in the behaviour of the solutions. [5]

(ii) Now consider the case $k = -3$.

(*A*) Find the general solution of the differential equation. [6]

(*B*) Find the particular solution corresponding to the curve drawn in part **(i)**. [2]

(*C*) Find the particular solution which remains bounded as $x \to \infty$. On the insert, sketch this solution on the tangent field. [3]

(iii) Now consider the case $k = 3$.

Does the tangent field suggest that any solution curves are unbounded? Verify your answer by considering the **form** of the general solution of the differential equation. [4]

Candidate Name	Centre Number	Candidate Number

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MEI STRUCTURED MATHEMATICS 2610/1

Differential Equations (Mechanics 4)

INSERT

Wednesday **23 JUNE 2004** Afternoon 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- This insert should be used in Question **4.**
- Write your Name, Centre Number and Candidate Number in the spaces provided at the top of this page and attach it to your answer booklet.

This insert consists of 2 printed pages.

HN/5

[Turn over

Jun04/erratum34

OXFORD CAMBRIDGE AND RSA EXAMINATIONS
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MEI STRUCTURED MATHEMATICS 2610/1

Differential Equations (Mechanics 4)

Wednesday **23 June 2004** Afternoon 1 hour 20 minutes

ERRATUM NOTICE

For the attention of the Examinations Officer

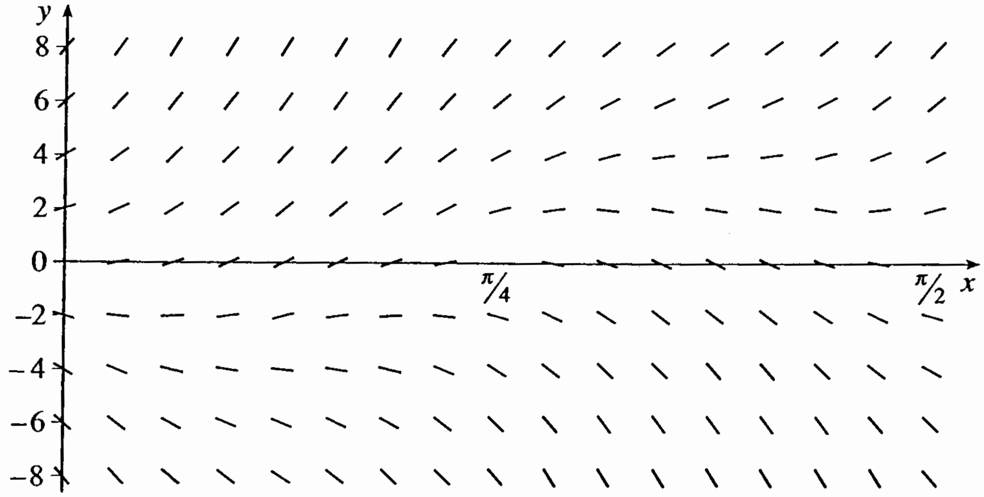
There is an error on the **Insert.**

Please inform all candidates before the start of the examination

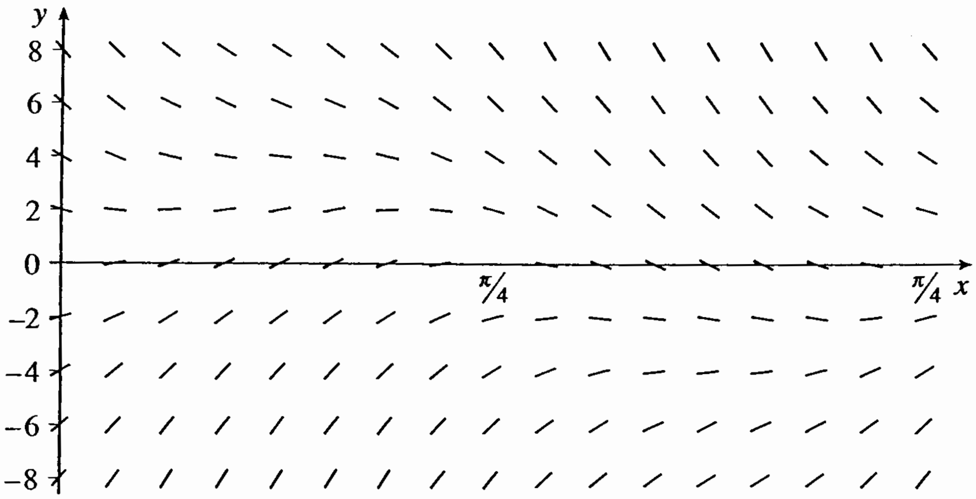
In the middle graph, the tangent field for $k = 3$, the second value on the x-axis **should be** $\pi/2$, **not** $\pi/4$.

Insert for question 4.

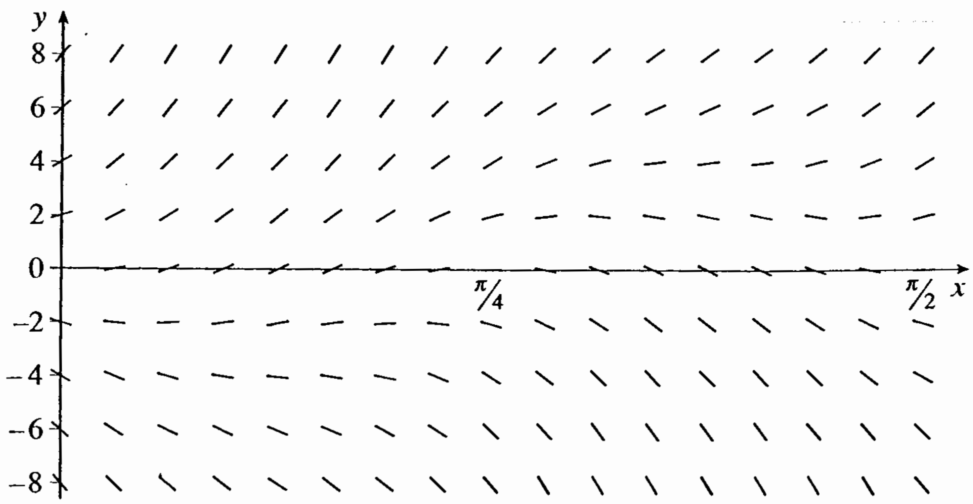
(i) Tangent field for $k = -3$.



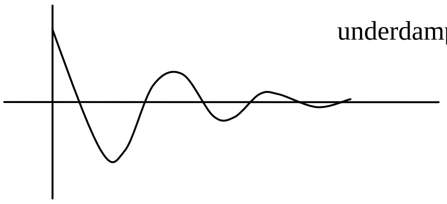
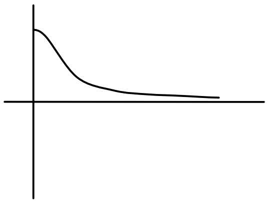
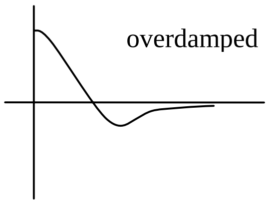
Tangent field for $k = 3$.



(ii) (*C*) Tangent field for $k = -3$.

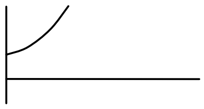
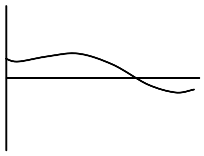
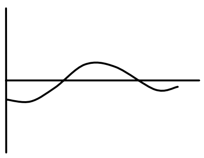


Mark Scheme

1(i)	equilibrium $\Rightarrow mg = ky \Rightarrow k = \dfrac{300g}{0.8} = 3675$	M1 E1	use of equilibrium	
				2
(ii)	N2L: mass × acc. = weight – resistance – upthrust	B1 B1	identify two terms identify all terms, including $m\ddot{y}$	
	(weight down so positive) upthrust up so negative	B1	allow upthrust only	
	resistance opposes motion so negative	B1		
	$(t = 0), y = 1, \dot{y} = 0$	B1		
				5
(iii)	underdamped 	B1	must be labelled and must decay	
	 or  overdamped	B1	must be labelled and must decay	
	discriminant = $300^2 - 4 \times 300 \times 3675 < 0 \Rightarrow$ underdamped	M1 A1	consider discriminant clearly reasoned argument	
				4
(iv)	$\ddot{y} + \dot{y} + 12.25y = 9.8$			
	PI $y = \dfrac{9.8}{12.25} = 0.8$	B1	correct PI (accept unsimplified)	
	CF $\lambda^2 + \lambda + 12.25 = 0$	M1	auxiliary equation	
	$\lambda = -\frac{1}{2} \pm 2\sqrt{3}\,i$	A1		
	$y = e^{-\frac{1}{2}t}\left(A\cos 2\sqrt{3}\,t + B\sin 2\sqrt{3}\,t\right)$	F1	CF for their roots	
	GS $y = 0.8 + e^{-\frac{1}{2}t}\left(A\cos 2\sqrt{3}\,t + B\sin 2\sqrt{3}\,t\right)$	F1	their CF + PI	
	$t = 0, y = 1 \Rightarrow A = 0.2$	M1	using y = 1 or their value if explicitly stated in (ii)	
	$\dot{y} = -\frac{1}{2}e^{-\frac{1}{2}t}\left(A\cos 2\sqrt{3}\,t + B\sin 2\sqrt{3}\,t\right)$ $+2\sqrt{3}\,e^{-\frac{1}{2}t}\left(-A\sin 2\sqrt{3}\,t + B\cos 2\sqrt{3}\,t\right)$	M1	differentiating (product rule)	
	$t = 0, \dot{y} = 0 \Rightarrow -\frac{1}{2}A + 2\sqrt{3}B = 0 \Rightarrow B = \dfrac{1}{20\sqrt{3}}$	M1	using $\dot{y}$ = 0 or their value if explicitly stated in (ii)	
	$y = 0.8 + 0.2\,e^{-\frac{1}{2}t}\left(\cos 2\sqrt{3}\,t + \dfrac{1}{4\sqrt{3}}\sin 2\sqrt{3}\,t\right)$	A1	cao	
				9

2(i)	$\frac{dx}{dt}+5x=0, \lambda+5=0 \Rightarrow x=Ae^{-5t}$	M1	or any valid method	
	$t=0, x=3 \Rightarrow A=3 \Rightarrow x=3e^{-5t}$	E1		
				2
(ii)	$\ddot{y}=\dot{x}-2\dot{y}+4\dot{z}$	M1	differentiate equation (2)	
	$=-15e^{-5t}-2\dot{y}+4(3e^{-5t}+3y-6z)$	M1	substitute for $\dot{z}$	
	$=-3e^{-5t}-2\dot{y}+12y-\frac{24}{4}\left(-3e^{-5t}+2y+\dot{y}\right)$	M1	substitute for z	
	$=15e^{-5t}-8\dot{y}$	M1	substitute for x and $\dot{x}$	
	$\ddot{y}+8\dot{y}=15e^{-5t}$	E1		
				5
(iii)	CF $\lambda^2+8\lambda=0 \Rightarrow \lambda=-8$ or 0	M1	solve auxiliary equation	
	$y=B+Ce^{-8t}$	A1		
	PI $y=ae^{-5t}$	B1	correct form	
	$\dot{y}=-5ae^{-5t}, \ddot{y}=25ae^{-5t} \Rightarrow 25a-40a=15$	M1		
	$\Rightarrow a=-1$	A1		
	GS $y=B+Ce^{-8t}-e^{-5t}$	F1	their CF + PI	
	initially $x=3, y=0, z=0 \Rightarrow \dot{y}=x-2y+4z=3-0+0=3$	E1	some working required	
	$t=0, y=0 \Rightarrow B+C-1=0$	M1	using condition on y	
	$t=0, \dot{y}=3 \Rightarrow -8C+5=3$	M1	using condition on $\dot{y}$	
	$\Rightarrow B=\frac{3}{4}, C=\frac{1}{4} \Rightarrow y=\frac{3}{4}+\frac{1}{4}e^{-8t}-e^{-5t}$	A1	cao	
				10
(iv)	$\frac{d}{dt}\left(x+3y+2z\right)=-5x+3(x-2y+4z)+2(x+3y-6z)=0$	M1	show derivative is zero (some working required)	
	so $x+3y+2z=\text{constant}=3$ (by initial conditions)	E1		
	$z=\frac{1}{2}\left(3-x-3y\right)=\frac{3}{8}\left(1-e^{-8t}\right)$	F1	follow their y	
				3

3(i)	$\ddot{\theta} \approx -16\theta$	B1		
	$\theta = A\cos 4t + B\sin 4t$	M1	general solution for SHM (or equivalent form)	
	$\dot{\theta}(0) = 0 \Rightarrow B = 0$			
	$\theta(0) = \alpha \Rightarrow A = \alpha$, so $\theta = \alpha\cos 4t$	A1		
	$\theta = 0 \Rightarrow t = \frac{1}{8}\pi$	B1		
				4
(ii)	$\dot{\theta}^2 \geqslant 0$	B1		
	$\Rightarrow \cos\theta \geqslant \cos\alpha$	M1	use of $\dot{\theta}^2 \geqslant 0$ in equation	
	$\Rightarrow \lvert\theta\rvert \leqslant \alpha$	E1		
	stationary when $\theta = \pm\alpha$	B1		
				4
(iii)	$\dot{\theta}(\frac{1}{4}\pi) = 0 \Rightarrow \theta_2 = \theta_1$	M1	consider effect in algorithm	
	$\Rightarrow \theta$ remains at $\frac{1}{4}\pi$	A1	conclude θ constant	
				2
(iv)	*either* symmetry of motion $\Rightarrow$ time from $\theta = 0$ to $\frac{1}{4}\pi$ is also T_1	B1	either reason	
	or $\dot{\theta} \neq 0 \Rightarrow \theta$ not constant when algorithm applied			
				1
(v)	t θ $\dot{\theta}$			
	0.37 0.7813 0.3042	M1		
	0.38 0.7843 0.1575	A1	$\theta(0.38)$	
	0.39 0.7859	M1		
		A1	$\theta(0.39)$	
	$0.7843 < \frac{1}{4}\pi < 0.7859$	M1	comparison	
	hence $T_1 = 0.38$ or 0.39	A1	accept either	
				6
(vi)	$\frac{d\theta}{dt} = -\sqrt{32\left(\cos\theta - \cos\frac{1}{4}\pi\right)}$			
	$\int_{\frac{1}{4}\pi}^{0} \frac{d\theta}{-\sqrt{32\left(\cos\theta - \cos\frac{1}{4}\pi\right)}} = \int_0^{T_1} dt$	M1	separate variables	
	$T_1 = \frac{1}{\sqrt{32}}\int_0^{\frac{1}{4}\pi} \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}} \approx \frac{1}{\sqrt{32}} \times 2.3012$	M1	attempt to get T_1 as an integral	
	$= 0.407$	A1		
				3

4(i)	$k = -3$	M1	attempt one curve	
		A1	curve through (0,1) roughly consistent with tangent field	
	increasing	B1	any reasonable description consistent with sketch	
	$k = 3$			
		A1	curve through (0,1) roughly consistent with tangent field	
	oscillating	B1	any reasonable description consistent with sketch	
				5
(ii) (*A*)	CF $y = A\mathrm{e}^{3x}$	B1		
	PI $y = a\cos 4x + b\sin 4x$	B1		
	$\dot{y} = -4a\sin 4x + 4b\cos 4x$	M1	differentiate and substitute	
	$4b - 3a = 0$ $-4a - 3b = 10$	M1	compare coefficients	
	$a = -\frac{8}{5}, b = -\frac{6}{5}$	A1		
	GS $y = A\mathrm{e}^{3x} - \frac{8}{5}\cos 4x - \frac{6}{5}\sin 4x$	F1	their CF + PI	
				6
(*B*)	$x = 0, y = 1 \Rightarrow A - \frac{8}{5} = 1 \Rightarrow A = \frac{13}{5}$	M1	using condition on y	
	$y = \frac{1}{5}\left(13\mathrm{e}^{3x} - 8\cos 4x - 6\sin 4x\right)$	F1	follow their GS	
				2
(*C*)	y bounded $\Rightarrow A = 0$	M1	choose constant to remove unbounded term	
	$y = -\frac{1}{5}\left(8\cos 4x + 6\sin 4x\right)$	F1	must be consistent with their GS and bounded	
		F1	bounded curve roughly consistent with tangent field and through y-intercept consistent with their solution	
				3
(iii)	No (tangent field does not suggest unbounded solutions)	B1		
	CF $y = A\mathrm{e}^{-3x}$	M1		
	PI $a\cos 4x + b\sin 4x$	M1		
	Both parts of GS are bounded	A1		
				4

Examiner’s Report

2610 Mechanics 4

General Comments

There were many good scripts and the standard of algebra was high. Questions 1 and 2 were the most popular choices, in particular question 2.

Candidates should take note that when asked for a solution to a differential equation with given conditions, that a particular solution is required, not a general solution. They should also take note that when asked for a particular solution, it should be stated explicitly, rather than just stating the values of the arbitrary constants.

Comments on Individual Questions

Q.1 Virtually all candidates were able to establish $k = 3675$, with only a few showing insufficient working for a given result. The responses to part (ii) were very variable. Some candidates gave good, clear answers, but many seemed unable to recognise a Newton's second law equation. Even for candidates who recognised the relevant forces, the justification of the signs was often vague, incorrect or omitted. The initial conditions were sometimes omitted, and when present, the value for y was frequently incorrect.

Although many candidates showed the distinction between an over-damped and an under-damped system, many candidates had problems with the over-damped system, showing it to oscillate, but decaying rapidly. A few candidates did not show the under-damped system to decay. The solution of the differential equation was often well done, although algebraic errors were relatively common. A few candidates made no use of the initial conditions to find the particular solution. Some candidates stated their complementary function in terms of complex exponentials and then derived the trigonometric version. They should be aware that once complex roots have been found, the relevant complementary function may be simply stated.

Q.2 Most candidates were able to solve the equation for x, although some candidates' solutions were very long. The elimination of z was often done well, but some candidates eliminated the derivative but were unsure how to then eliminate the z term. The solution of the equation was often done well, although a significant minority did not know what to do with the zero solution of the auxiliary equation, some omitting a term in the complementary function, and others omitting the zero root and then using the other root as a repeated root. Only a handful of candidates used the neat method of regarding the equation as a first order equation in $\mathrm{d}y/\mathrm{d}t$, and then integrating the solution to get y. Few candidates showed that the derivative of y was 3 initially, but most were able to use it successfully. In the final part of the question, many candidates struggled to justify the given equation, but most used it correctly to find z.

Q.3 Candidates rarely had problems stating the approximate differential equation, but some found the solution difficult, either getting the wrong roots to the auxiliary equation, or being unable to find the arbitrary constants. In part (ii), justifications of the inequality were often lacking in detail for a 'show that' question. Some candidates also wrongly decided that to get $\theta \leqslant \alpha$ they needed $\cos\theta \leqslant \cos\alpha$ in the previous line, and changed a correct line to an incorrect one!

The numerical solution was often done well, but some candidates lost all the marks because they had two incorrect values for θ and no working. If using a tabular method, candidates should at least include values of the derivative, so that they are providing some evidence that the algorithm is being used. In the final part only a minority of candidates realised that they could separate variables and hence use the given numerical integral to estimate the time.

Q.4 Sketches on the tangent fields were generally good, but some candidates seemed to ignore the direction lines unless their curve intersected one. Descriptions of the solutions were often vague and some were expressed in very un-mathematical language (e.g. “the curve goes up a bit and then down a bit…”), but were generally adequate. The solution of the differential equation was often well done using a complementary function and particular integral. Candidates using the integrating factor method produced a difficult integral, but many then switched method and successfully completed the solution. The particular solution passing through (0, 1) was usually correct but the bounded particular solution was found more difficult. In the final part, candidates’ responses were sometimes vague or incomplete when justifying why solutions would always be bounded. Many ignored the request to consider what the tangent field suggested. The request to consider the general solution was often ignored and candidates considered only the complementary function. Others found the general solution explicitly, whereas only the form was required.

Coursework: Differential Equations (Mechanics 4)

About 25% of centres had their work moderated downwards, which is a little higher than usual.

“Aeroplane Landing” was popular, as usual. The most common problems here were not acknowledging the source of data and not commenting on the accuracy of the data used. Only the very best scripts considered a variation in the parameters. There is a tendency to model and revise the first part of the motion before the brakes are applied and then only use the form of air resistance from the most successful model for their single model of the motion after the brakes are applied. Candidates should be encouraged to model the complete motion for all their assumptions about the type of resistance as they may well decide that while one assumption about the type of resistance gives the best match at high speeds another gives the better match at slower speeds.

For investigations such as “Cascades”, “Paper Cups”, etc, using an initial differential equation which only involves a constant is really too trivial to be considered as a first model. The differential equation should reflect the syllabus for this module. Also, the work on “Cascades” **must** be centred on the flow in the middle container, otherwise again the work is trivial.

Care must be taken to avoid circular arguments by using the data from the experiment to predict results in the experiment. For example, in “Paper Cups”, the parameters could be deduced from an experiment with one cup and then used to predict the results for 2, 3 or more cups.

Although it may be difficult in some experiments, sufficient repetitions are necessary for variation to be considered adequately.

As a general point, the discussion of the assumptions is an essential part of the modeling process and a simple list is not sufficient to meet the criteria in Domain 1. Also, when a second model is proposed, there does need to be discussion about modifying the assumptions and/or justify the model; often this was absent.