OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS **2610/1**

Differential Equations (Mechanics 4)

Tuesday **17 JUNE 2003** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- Take $g = 9.8 \text{ m s}^{-2}$ unless otherwise instructed.
- The total number of marks for this paper is 60.

This question paper consists of 4 printed pages.

1 A solution is sought to the differential equation

$$\ddot{y} + 4\dot{y} + 4y = 2e^{-2t}.$$

(i) Find the complementary function. [3]

(ii) Explain why an expression of either of the forms ae^{-2t} or ate^{-2t} cannot be a particular integral of the differential equation. State a correct form for the particular integral and hence show that the general solution is

$$y = (A + Bt + t^2)e^{-2t},$$

where A and B are arbitrary constants. [6]

(iii) It is given that $y = 0$ when $t = 1$ and that $y = 0$ when $t = 3$. Calculate the constants A and B and write the particular solution in factorised form. [5]

(iv) By reference to the previous answer, write down the particular solution in the cases

(*A*) $y = 0$ when $t = 2$ and $y = 0$ when $t = 7$,

(*B*) $y = 0$ only when $t = 1$.

Write down an example of a particular solution which is never zero. Justify your answer. [6]

2 A radioactive element X decays into the radioactive element Y which decays into the stable element Z. The decay can be modelled by the differential equations

$$\frac{dx}{dt} = -0.1x, \qquad (1)$$

$$\frac{dy}{dt} = 0.1x - 0.2y, \qquad (2)$$

$$\frac{dz}{dt} = 0.2y, \qquad (3)$$

where x, y and z are the masses (in milligrams) of X, Y and Z respectively at time t seconds.

When $t = 0$, the mass of X is 25 mg, and there is no Y or Z present.

(i) Solve equation (1) to find x in terms of t. Sketch a graph of your solution. [4]

(ii) Use your solution for x to solve equation (2) to find y in terms of t. [7]

(iii) Calculate the range of times for which the amount of Y is increasing. Sketch a graph of the mass of Y against time, showing the maximum value. [4]

(iv) Without solving equation (3), show that $x + y + z$ is constant. Hence find z in terms of t, and verify that this satisfies equation (3) and the initial conditions. [5]

3 The solution to the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{\frac{1}{25}+x^2y^2}$$

is to be investigated, first by means of a tangent field and then numerically.

(i) Show that $x = 0$ is an isocline. Show that all other isoclines are of the form $y=\frac{a}{x}$. [4]

Solution curves are to be drawn through the points (–2, 0) and (0, –1).

(ii) Sketch sufficient isoclines in order to produce the tangent field and hence sketch the two curves. [8]

A numerical solution is now sought using Euler's method. The algorithm is given by $x_{r+1}=x_r+h$, $y_{r+1}=y_r+h\mathrm{f}(x_r, y_r)$, where $\mathrm{f}(x,y)=\frac{\mathrm{d}y}{\mathrm{d}x}$.

(iii) Apply the algorithm for two equal steps, starting from the point (0, 0), to estimate the value of y when $x = 0.4$. If the estimate is recalculated with a smaller step length, will the estimate be less or greater? Explain your reasoning. [8]

TURN OVER for Q 4

4 Three differential equations are being studied for $x > 0$. The first is

$$\frac{dy}{dx} + \frac{y}{x} = e^x. \qquad (1)$$

(i) Use the integrating factor method to find the general solution of equation (1). [6]

(ii) Given that $y = 1$ when $x = 1$, show that the particular solution is $y = \frac{(x-1)e^x + 1}{x}$.

Use the approximation $e^x \approx 1 + x$, for small x, to describe the behaviour of y as x tends to zero.

Sketch a graph of the solution. [6]

The other two differential equations are

$$\frac{dy}{dx} + y = e^x, \qquad (2)$$

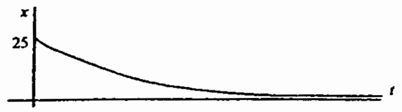
$$\frac{dy}{dx} + ye^x = e^x. \qquad (3)$$

(iii) Use a method other than the integrating factor method to find the general solution of equation (2). [5]

(iv) Equation (3) can be solved by a different method from equations (1) and (2). Use this method to find the general solution of equation (3). [3]

Mark Scheme

1(i)	$\alpha^2+4\alpha+4=0$	M1		
	$\alpha=-2$ (repeated)	A1		
	CF $y=(A+Bt)e^{-2t}$	F1	CF for their roots (y in terms of t)	
				3
(ii)	They are the same form as the CF so satisfy homogeneous equation, hence will not satisfy the non-homogenous equation.	B1	justifies that it will not satisfy DE (may use substitution)	
	PI $y=at^2e^{-2t}$	B1	correct form	
	$(2a-8at+4at^2)e^{-2t}+4(2at-2at^2)e^{-2t}+4at^2e^{-2t}=2e^{-2t}$	M1	differentiate twice and substitute	
	$2ae^{-2t}=2e^{-2t}$	M1	compare coefficients	
	$a=1$	A1		
	$y=(A+Bt+t^2)e^{-2t}$	E1		
				6
(iii)	$(A+B+1)e^{-2}=0$	B1		
	$(A+3B+9)e^{-6}=0$	B1		
		M1	solving	
	$A=3, B=-4$	A1		
	$y=(3-4t+t^2)e^{-2t}=(t-1)(t-3)e^{-2t}$	A1	must be factorised (cao)	
				5
(ii)	(A) $y=(t-2)(t-7)e^{-2t}$	M1 A1		
	(B) $y=(t-1)^2e^{-2t}$	M1 A1		
	e.g. $y=(t^2+1)e^{-2t}$	B1	quadratic with negative discriminant $\times e^{-2t}$	
	e.g. $t^2+1>0, e^{-2t}>0 \Rightarrow y>0$	B1	showing never zero	
				6

Part	Working	Mark	Comment	Total
2(i)	solve by separating variables or CF	M1		
	$x = Ae^{-0.1t}$	A1	accept answer only	
	$t = 0, x = 25 \Rightarrow x = 25e^{-0.1t}$	A1		
		B1	sketch with initial condition and tending to 0	4
(ii)	$\frac{dy}{dt} + 0.2y = 2.5e^{-0.1t}$			
	$\alpha + 0.2 = 0$	M1		
	CF $y = Be^{-0.2t}$	A1		
	PI $y = ae^{-0.1t}$	B1	correct form	
	$-0.1ae^{-0.1t} + 0.2ae^{-0.1t} = 2.5e^{-0.1t} \Rightarrow a = 25$	M1	differentiate, substitute and compare coefficients	
	$y = 25e^{-0.1t} + Be^{-0.2t}$	A1		
	$t = 0, y = 0 \Rightarrow 25 + B = 0$	M1		
	$y = 25(e^{-0.1t} - e^{-0.2t})$	A1		7
(iii)	$\frac{dy}{dt} > 0 \Rightarrow -2.5e^{-0.1t} + 5e^{-0.2t} > 0 \Rightarrow e^{0.1t} < 2$	M1		
	$\Rightarrow t < 10\ln 2 \approx 6.93$	A1		
	$y_{max} = 25(2^{-1} - 2^{-2}) = 6.25$			
		B1 B1	general shape through origin and maximum labelled	4
(iv)	$\frac{d}{dt}(x + y + z) = -0.1x + (0.1x - 0.2y) + 0.2y = 0$			
	$\Rightarrow x + y + z =$ constant	E1		
	hence $x + y + z = 25$	M1		
	$\Rightarrow z = 25(1 - 2e^{-0.1t} + e^{-0.2t})$	A1		
	$\frac{dz}{dt} = 5e^{-0.1t} - 5e^{-0.2t} = 0.2y$	E1	verify satisfies DE	
	$t = 0 \Rightarrow z = 25(1 - 2 + 1) = 0$	E1	verify satisfies conditions	5

3(i)	$x = 0 \Rightarrow \frac{dy}{dx} = \sqrt{\frac{1}{25}} = \frac{1}{5}$ so $x = 0$ is an isocline	B1		
	$\frac{dy}{dx} = \text{constant} \Rightarrow \sqrt{\frac{1}{25} + x^2y^2} = \text{constant} \Rightarrow (xy)^2 = \text{constant}$	M1		
	$\Rightarrow xy = \text{constant}$	A1		
	$\Rightarrow y = \frac{a}{x}$ if $x \neq 0$	E1		4
(ii)		M1		
		A1	isoclines	
		M1		
		A1	direction indicators	
		M1		
		A1	curve through (0,1)	
		M1		
		A1	curve through (-2,0)	8
(iii)	x y y'	M1	attempt algorithm once	
	0 0 0.2	A1	$y(0.2)$	
	0.2 0.04 0.20016	M1	$y'(0.2)$ evaluated correctly to at least 4dp	
	0.4 0.080032	M1	attempt algorithm for second step	
		A1	$y(0.4)$ (not 0.08 unless working shown)	
	gradient increases away from origin	B1		
	more steps mean gradient increases more	B1		
	so value of y greater	B1		8

4(i)	$I = \exp(\int \frac{1}{x}\,\mathrm{d}t)$	M1		
	$= \exp(\ln x) = x$	A1		
	$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = xe^x$	M1	multiply	
	$xy = \int xe^x \mathrm{d}x$	M1	integrate	
	$= xe^x - \int e^x \mathrm{d}x = xe^x - e^x + c$	A1		
	$y = \dfrac{xe^x - e^x + c}{x}$	A1		6
(ii)	$x = 1, y = 1 \Rightarrow 1 = (e - e + c)/1 \Rightarrow c = 1$	M1		
	$y = \dfrac{(x-1)e^x + 1}{x}$	E1		
	$e^x \approx 1 + x \Rightarrow y \approx \dfrac{(x-1)(1+x)+1}{x}$	M1		
	$= x$	A1		
	so $y \to 0$ as $x \to 0$	A1		
		B1		6
(iii)	complementary function + particular integral	B1	correctly identify method	
	CF $\alpha + 1 = 0 \Rightarrow \alpha = -1$	M1		
	$y = Ae^{-x}$	A1		
	PI $y = ae^x \Rightarrow ae^x + ae^x = e^x \Rightarrow a = \frac{1}{2}$	M1	correct form of PI, differentiate, substitute, compare coefficients	
	GS $y = Ae^{-x} + \frac{1}{2}e^x$	A1		5
(iv)	$\frac{\mathrm{d}y}{\mathrm{d}x} = e^x(1-y)$	M1	factorise	
	$\int \frac{\mathrm{d}y}{1-y} = \int e^x \mathrm{d}x$	M1	separate variables and attempt integration	
	$-\ln\lvert 1-y\rvert = e^x + c \Rightarrow y = 1 - Ae^{-e^x}$	A1		3

Examiner’s Report

2610 Differential Equations

General Comments

The performance of the candidates in this examination was very good with substantial numbers scoring high marks. They were well prepared and answered questions confidently and quickly. It was very noticeable that careless errors this time were rather scarce – an unusual but welcome result.

Comments on Individual Questions

Q.1 In this question on the solution of the second order linear differential equation, the complementary function in part (i) was routinely found correctly by the vast majority of candidates. The explanations why neither ae^{-2t} nor ate^{-2t} can be in the particular integral were not too good, but were balanced by the completion of the general solution anyway. Finding particular solutions given initial conditions was well done. The final part was much more tenuous in quality with many unable to suggest a solution which was never zero.

(i) $y=(A+Bt)e^{-2t}$; (iii) $y=(t-1)(t-3)e^{-2t}$;
(iv)(*A*) $y=(t-2)(t-7)e^{-2t}$, (*B*) $y=(t-1)^2e^{-2t}$.

Q.2 On the whole the solutions of the three equations representing radioactive decays of elements X to Y and then to Z were also presented well. As in question 1, the difficulty came in the final part, in this case to explain why $x+y+z$ is constant without resorting to solving the equations. The answers often a demonstration that z satisfied the initial conditions.

(i) $x=25e^{-0.1t}$; (ii) $y=25(e^{-0.1t}-e^{-0.2t})$; (iii) $t<10\ln 2$; (iv) $z=25(1-2e^{-0.1t}+e^{-0.2t})$.

Q.3 This question on isoclines was without doubt the least popular question on the paper and the worst answered. Very few candidates indeed drew any isoclines at all in part (ii) although a (usually) rudimentary attempt was made at the tangent field. The numerical solutions were often poor handicapped by layout A systematic approach to such problems yields high dividends.

(iii) 0.080032.

Q.4 This question was extremely well answered with the appropriate method chosen to solve the three different equations presented. The occasional omission of the integration constant caused problems for some but most had little trouble. In part (ii), finding the behaviour of the solution near zero and the sketch were done very well. Candidates were only required to observe that y tends to zero as x tends to zero for which the linear approximation for e^x is sufficient. It is acknowledged that the quadratic approximation is required to give the correct linear approximation to y. However on their sketch candidates were only required to indicate that the solution approaches the origin, but not *how* it approaches the origin.

(i) $y=\frac{1}{x}(xe^x-e^x+c)$; (iii) $y=Ae^{-x}+\frac{1}{2}e^x$; (iv) $y=1-Ae^{-e^x}$.