GENERAL CERTIFICATE OF EDUCATION EXAMINATION

SUMMER 1970

Special Paper

PURE MATHEMATICS

Three hours

Answer EIGHT questions.

1. Show that if $b^2 \le 4a^3$ all the roots of the equation

$$x^3 - 3ax + b = 0$$

are real, and hence find the range of values of k for which the equation

$$4x^3 + 24x^2 + 45x + k = 0$$

has three real roots. Solve this equation for each of the values of k for which two roots are equal.

2. (i) Prove the inequalities

(*a*) $e^x \ge x + 1$, for all values of x,

(*b*) $x - 1 \ge \ln x$, for $x > 0$.

(ii) If the sum of the positive numbers a, b, c is 3, find the range of the possible values of $(a^2 + b^2 + c^2)$.

3. (i) If s_n denotes the sum of the first n terms of the series in which the rth term is $(r^2 + r - 1)/(r^2 + r)$, show that s_n lies between $(n - 1)$ and n.

(ii) Find the sum of the infinite series in which the nth term is

(*a*) $(n - 1)^3/n!$

(*b*) $1/(2n^2 + n)$.

4. The point P represents the complex number z in the Argand diagram. Find the locus of the point representing the number $2z/(z-1)$ when P moves round the circle $|z| = 1$.

Describe the locus defined by each of the following equations, and illustrate each locus in an Argand diagram.

(*a*) $|z+1|^2 + |z-1|^2 = 4$,

(*b*) $|z+\mathrm{i}| + |z-\mathrm{i}| = 3$,

(*c*) $\arg(z-1) = \arg(z+1)$.

5. If x is not a multiple of π and n is a positive integer, show that

$$\sin x + \sin 3x + \ldots + \sin(2n-1)x = \sin^2 nx \operatorname{cosec} x,$$

and find the sum of the series.

Evaluate the integrals

(*a*) $\displaystyle\int_0^{2\pi} \sin^2 6x \operatorname{cosec} x \;\mathrm{d}x$,

(*b*) $\displaystyle\int_0^{2\pi} \sin^4 6x \operatorname{cosec}^2 x \;\mathrm{d}x$,

6. (i) Sketch the curve $y = \mathrm{e}^{-2x}\sin x$, and show that for any positive integer n

$$\int_0^{2n\pi} \mathrm{e}^{-2x}\sin x \;\mathrm{d}x < 1/5 < \int_0^{(2n+1)\pi} \mathrm{e}^{-2x}\sin x \;\mathrm{d}x.$$

(ii) By using the substitution $x = 1/(y-1)$, or otherwise, evaluate the integral

$$\int_1^3 \frac{\mathrm{d}x}{(x+1)\sqrt{(x^2+x)}}.$$

7. Obtain the equation of the chord PQ of the ellipse

$$x^2/a^2 + y^2/b^2 = 1,$$

given that the coordinates of its mid-point M are (h, k).

Find the equation satisfied by the coordinates of M

(a) if PQ passes through the point (a, b),

(b) if the perpendicular bisector of PQ passes through the point (a, b).

8. The gradient m of the chord PQ of the hyperbola $xy = c^2$ is constant and positive. Show that there are two fixed points through which the circle on PQ as diameter passes for all positions of PQ.

Show also that if the chord RS is perpendicular to PQ, the circle on RS as diameter cuts orthogonally the circle on PQ as diameter.

9. Obtain the conditions for the line

$$(x - a)/l = (y - b)/m = (z - c)/n$$

to lie in the plane $Ax + By + Cz = D$.

Find the coordinates of the point N, the foot of the perpendicular from the origin to the plane $2x + y + 2z = 27$.

Find also the equations of the lines in this plane which pass through N and which make an angle of 60° with the line $x - y = 3$, $z = 6$.

10. Show that at any point (h, k) on the curve $(x + y)^3 = 9xy$ (except the origin) the gradient is $(2hk - k^2)/(h^2 - 2hk)$, and find the equations of the tangents to the curve which are parallel to the x-axis.

Find the equation of the locus of the mid-points of chords of the curve which are parallel to the line $x + y = 0$.