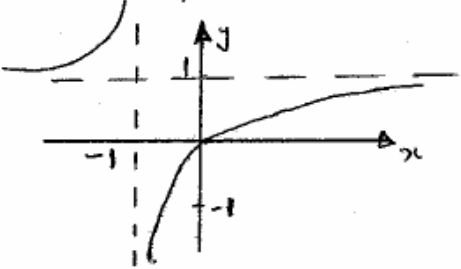


June 2006
9801 Advanced Extension Award
Mark Scheme

Question Number	Scheme	Marks
1 (a)	$(1-y)^{-2} = 1 + 2y + 3y^2 + 4y^3 + \dots$	B1 (1)
1 (b)	$S = 1 + 2\left(\frac{x}{1+x}\right) + 3\left(\frac{x}{1+x}\right)^2 + \dots$ <p style="text-align: right;">Identify $y = \frac{x}{1+x}$</p> $\Rightarrow S = 1 + 2y + 3y^2 + \dots$ $= \left(1 - \frac{x}{1+x}\right)^{-2}$ $= \frac{1}{(1+x)^2}, \text{ so } \underline{a=1, n=2}$	M1 A1 M1, A1 (4)
1 (c)	<p>Need $\left \frac{x}{1+x}\right < 1$ correct condition</p>  <p style="text-align: right;">Critical value is $\frac{x}{x+1} = -1$</p> $\Rightarrow x = -\frac{1}{2}$ $\therefore \underline{x > -\frac{1}{2}}$	B1 M1 A1 (3)

Question Number	Scheme	Marks
<u>2</u>	$(\sin 2\theta - \sqrt{3} \cos 2\theta) \left[\frac{2 \cos 2\theta}{\sin \theta + \cos \theta} - \sqrt{6} \right] = 0$ $\sin 2\theta - \sqrt{3} \cos 2\theta = 0 \Rightarrow \tan 2\theta = \sqrt{3} \text{ or } \sin(2\theta - 60^\circ) = 0$ $\Rightarrow 2\theta = 60^\circ, 240^\circ, 420^\circ, 600^\circ$ $\theta = 30^\circ, 120^\circ, 210^\circ, 300^\circ$ $\frac{2 \cos 2\theta}{\sin \theta + \cos \theta} - \sqrt{6} = 0 \Rightarrow \frac{2(\cos^2 \theta - \sin^2 \theta)}{\sin \theta + \cos \theta} = \sqrt{6} \quad (\text{use } \cos 2\theta = \dots)$ $= 2 \frac{(\cos \theta - \sin \theta)(\cos \theta + \sin \theta)}{(\cos \theta + \sin \theta)} = \sqrt{6} \quad (\text{Factor + cancel})$ $(\because \cos \theta + \sin \theta \neq 0)$ $\therefore \cos(\theta + 45^\circ) = \frac{\sqrt{6}}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2}$ $\theta + 45^\circ = (30^\circ), 330^\circ, 390^\circ$ $\theta = 285^\circ, 345^\circ$	M1 (factor) A1 M1 A1 M1 M1, A1 R cos(45) one correct value for (θ+45°) M1 A1 (10)

Qu. 3.

(a)

$$\log_y x = z \quad \therefore x = y^z$$

$$\therefore y = x^{1/z} \Rightarrow \log_x y = \frac{1}{z} = \frac{1}{\log_y x}$$

or $\log_x x = \log_x y^z = z \log_x y = 1$ (Answer)

$$\therefore \log_x y = \frac{1}{\log_y x} \quad (2)$$

M1
(out of logs)

A1 c.s.o
(2)

(b)

$$\log_x y = \log_y x = \frac{1}{\log_x y} \Rightarrow (\log_x y)^2 = 1$$

$$\therefore \log_x y = \pm 1$$

M1

$$\log_x y \neq 1 \quad \because x \neq y \quad \therefore \log_x y = -1$$

$$\therefore y = \frac{1}{x}$$

A1 c.s.o
(2)

(c)

First equation $\Rightarrow y = \frac{1}{x}$

Second equation $\Rightarrow \log_x (x - \frac{1}{x}) = \log_{\frac{1}{x}} (x + \frac{1}{x}) = z$

M1

$$\therefore x^z = x - \frac{1}{x} \quad ; \quad (\frac{1}{x})^z = x + \frac{1}{x}$$

$$\therefore x^z (\frac{1}{x})^z = 1 = (x - \frac{1}{x})(x + \frac{1}{x}) \quad (\text{eliminate } z)$$

M1

$$\Rightarrow x^2 = (x^2 - 1)(x^2 + 1)$$

$$\Rightarrow x^4 - x^2 - 1 = 0 \quad (\text{quadratic})$$

M1, A1

$$x^2 = \frac{1 \pm \sqrt{5}}{2}$$

$$x^2 > 0, \text{ hence } x^2 = \frac{1 + \sqrt{5}}{2} \quad (\text{quad. formula } + x^2 > 0)$$

M1

$$x = \sqrt{\frac{1 + \sqrt{5}}{2}} \quad (\text{value } + \text{ve root})$$

A1

$$y = \frac{1}{x} = \sqrt{\frac{2}{1 + \sqrt{5}}} \quad (\text{or } \sqrt{\frac{\sqrt{5} - 1}{2}})$$

A1 ✓

(7)

Qn 4

(a)

$$(x+4)^2 + (y-7)^2 = 13 \quad \text{and} \quad y = mx$$

$$\therefore (x^2 + 8x + 16) + (m^2x^2 - 14mx + 49) = 13$$

$$(1+m^2)x^2 + (8-14m)x + 52 = 0$$

(3 b-grad) A1

Touche, so " $b^2 = 4ac$ " $(8-14m)^2 = 4 \times 52 \times (1+m^2)$

$$(4-7m)^2 = 52 + 52m^2$$

$$\therefore \underline{3m^2 + 56m + 36 = 0} \quad *$$

M1 ($b^2=4ac$)

A1 also

(4)

(b)

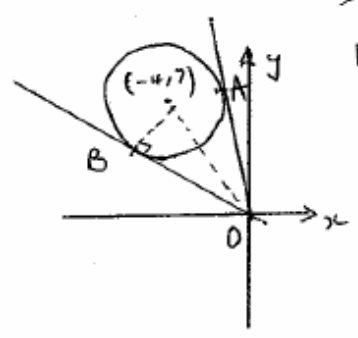
$$(3m+2)(m+18) = 0$$

$$\therefore m = -2/3 \quad \text{or} \quad -18$$

(both m)

M1

A1



Let A or B be (x, y)

$$\text{then } (x^2 + y^2) + 13 = 4^2 + 7^2 = 65$$

$$x^2 + y^2 = 52$$

M1, A1

$$y = -2/3x \Rightarrow \frac{13}{9}x^2 = 52 \Rightarrow x = \pm 6$$

From the configuration

$$x_B = -6 \therefore y_B = +4 \therefore \underline{B \text{ is } (-6, 4)}$$

} M1, A1

$$y = -18x \Rightarrow 325x^2 = 52 \Rightarrow x^2 = \frac{4}{25}$$

$$\text{Again } x < 0 \text{ for A} \therefore x_A = -\frac{2}{5}; y_A = \frac{36}{5}$$

$$\underline{A = (-\frac{2}{5}, \frac{36}{5})}$$

M1, A1

(8)

(c)

Situation is a translation of problem in (b) by $\begin{pmatrix} 4 \\ -7 \end{pmatrix}$

M1

$$\text{So } P, Q \text{ are } \begin{pmatrix} -6 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ -7 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\frac{2}{5} \\ \frac{36}{5} \end{pmatrix} + \begin{pmatrix} 4 \\ -7 \end{pmatrix}$$

$$= \underline{\underline{(-2, -3)}} \quad \text{and} \quad \underline{\underline{(\frac{18}{5}, \frac{1}{5})}}$$

A1 (either)

(2)

Qns
(a)

If L_1, L_2 intersect, then $r_1 = r_2$

$$\begin{aligned} i &\Rightarrow -2 + 3\lambda = 11.5 + 7\mu \Rightarrow -13.5 + 3\lambda = 7\mu \quad (1) \\ j &\Rightarrow -11.5 - 4\lambda = 3 + 8\mu \Rightarrow -14.5 - 4\lambda = 8\mu \quad (2) \\ k &\Rightarrow -\lambda = 8.5 - 11\mu \Rightarrow 8.5 + \lambda = 11\mu \quad (3) \end{aligned}$$

Solve any pair of these

(1) & (2) $\Rightarrow \lambda = 1/8, \mu = -15/8$

(1) & (3) $\Rightarrow \lambda = 8, \mu = 3/2$

(2) & (3) $\Rightarrow \lambda = -35/8, \mu = 3/8$

Check in third equation \Rightarrow inconsistent

Hence L_1, L_2 do not intersect

(Any one eqn) M1
(Attempt to solve 2) M1

(soln for any 2 eqn) A1

(check) M1

(comment) A1 (5)

(b)

$\begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 6 - 4 - 2 = 0 \therefore \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \perp L_1$ (A correct product) M1

$\begin{pmatrix} 7 \\ 8 \\ -11 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 14 + 8 - 22 = 0 \therefore \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \perp L_2$ (both) A1 (2)

(c)

$\vec{AB} = \begin{pmatrix} 7\mu + 11.5 - (-2 + 3\lambda) \\ 8\mu + 3 - (-11.5 - 4\lambda) \\ -11\mu + 8.5 - (-\lambda) \end{pmatrix} = \begin{pmatrix} 7\mu - 3\lambda + 13.5 \\ 8\mu + 4\lambda + 14.5 \\ -11\mu + \lambda + 8.5 \end{pmatrix}$ (Form \vec{AB}) M1

$\vec{AB} \parallel \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \therefore \underline{“i”} = \underline{“k”} \Rightarrow 7\mu - 3\lambda + \frac{27}{2} = -11\mu + \lambda + \frac{17}{2}$ A1
 $\Rightarrow 18\mu - 4\lambda + 5 = 0$ (4) A1

$\underline{“2j”} = \underline{“i”} \Rightarrow 16\mu + 8\lambda + 29 = 7\mu - 3\lambda + 13.5$ M1
 $\Rightarrow 9\mu + 11\lambda + 15.5 = 0$ (5) A1

[Note $\underline{“2j”} = \underline{“k”} \Rightarrow 27\mu + 7\lambda + 20.5 = 0$]

Solve (4) & (5) $\Rightarrow \underline{\lambda = -1; \mu = -1/2}$ (solving) M1
 (or any 2 eqns)

$\therefore \underline{OA} = \begin{pmatrix} -5 \\ -7.5 \\ 1 \end{pmatrix}$

$\underline{OB} = \begin{pmatrix} 8 \\ -1 \\ 14 \end{pmatrix}$

A1, A1

(8)

Q6

(a) $x=1; y = \sin(\ln 1) = \sin 0 = 0$
 $\therefore P = (1, 0)$ and P lies on C

B1
 a.s.o (i)

(b) $y' = \frac{1}{x} \cos(\ln x)$

M1, A1

$y' = 0$ at $Q \therefore \cos(\ln x) = 0 \therefore \ln x = \pi/2$
 $x = e^{\pi/2}$

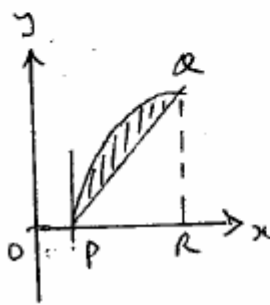
M1

$\therefore Q = (e^{\pi/2}, \sin(\ln e^{\pi/2}))$
 $= (e^{\pi/2}, 1)$

A1

A1 (5)

(c)



Area = $\int_1^{e^{\pi/2}} \sin(\ln x) dx$ - Area ΔPQR (correct approach) M1

Area $\Delta PQR = \frac{1}{2} \times 1 \times (e^{\pi/2} - 1)$ B1

For integral; let $\ln x = u \therefore x = e^u$ (subst) M1
 $\frac{1}{x} dx = du \therefore dx = e^u du$

$I = \int_0^{\pi/2} \sin u \cdot (e^u du)$ (int) A1

$= [e^u \sin u]_0^{\pi/2} - \int e^u \cos u du$ (limits) M1
 (part) M1

$= e^{\pi/2} - [e^u \cos u]_0^{\pi/2} - \int e^u \sin u du$ (second part) M1

$\therefore 2I = e^{\pi/2} + 1$

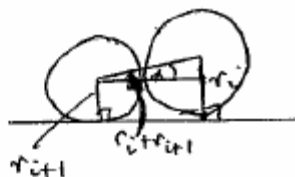
$I = \frac{1}{2}(1 + e^{\pi/2})$ (I) A1

$\therefore \text{Area} = \frac{1}{2}(1 + e^{\pi/2}) - \frac{1}{2}(-1 + e^{\pi/2}) = \underline{\underline{1}}$ A1

(9)

Qn 7.

(a)



Appropriate figure

$$\Rightarrow \sin \alpha = \frac{r_2 - r_1 \sin \alpha}{r_2 + r_1 \sin \alpha} \quad (\text{exp. for } \sin \alpha)$$

$$\therefore (r_2 + r_1 \sin \alpha) \sin \alpha = r_2 - r_1 \sin \alpha$$

$$\therefore r_1 \sin \alpha (1 + \sin \alpha) = r_2 (1 - \sin \alpha)$$

$$\therefore \text{ratio of radii} = \frac{1 - \sin \alpha}{1 + \sin \alpha} \quad (\text{*)} \quad (=r)$$

M1

M1, A1

($\frac{r_1}{r_2}$) M1

Also

(5)

(b)

$$\text{Total area} = \pi R^2 + \pi r_2^2 + \pi r_3^2 + \dots$$

$$= \pi R^2 (1 + r^2 + r^4 + \dots) \quad (\text{correct "r"})$$

$$= \frac{\pi R^2}{1 - r^2} = \pi R^2 \frac{1}{1 - \left(\frac{1 - \sin \alpha}{1 + \sin \alpha}\right)^2}$$

$$= \frac{\pi R^2 (1 + \sin \alpha)^2}{(1 + \sin \alpha)^2 - (1 - \sin \alpha)^2} = \frac{\pi R^2 (1 + \sin \alpha)^2}{4 \sin \alpha}$$

B1

M1

A1

(3)

(c)

$$\text{Required area} = 2 \times \text{Area } \triangle POA + \text{Area major sector } AOB - \text{Area found in (b)}$$

$$\text{Area } \triangle POA = \frac{1}{2} R (R \cot \alpha)$$

$$\angle POA = \frac{\pi}{2} - \alpha \quad \therefore \text{angle of major sector } AOB = \pi + 2\alpha$$

$$\therefore \text{Area sector } AOB = \frac{1}{2} R^2 (\pi + 2\alpha)$$

$$\therefore \text{Required area} = R^2 \left(\cot \alpha + \frac{\pi}{2} + \alpha - \frac{\pi}{4} \left(\frac{1 + 2 \sin \alpha + \sin^2 \alpha}{\sin \alpha} \right) \right)$$

$$= R^2 \left(\alpha + \cot \alpha - \frac{\pi}{4} \csc \alpha - \frac{\pi}{4} \sin \alpha \right) \quad (\text{*)}$$

M1

B1

M1

A1

(*) Also

(5)

Qn 7
(Cont.)

$$(d) \frac{dS}{d\alpha} = R^2 \left(1 - \operatorname{cosec}^2 \alpha + \frac{\pi}{4} \operatorname{cosec} \alpha \cot \alpha - \frac{\pi}{4} \cos \alpha \right)$$

$$= R^2 \left(-\cot^2 \alpha + \frac{\pi}{4} \frac{\cos \alpha}{\sin^2 \alpha} - \frac{\pi}{4} \cos \alpha \right)$$

$$= R^2 \left(-\cot^2 \alpha + \frac{\pi}{4} \cos \alpha (\operatorname{cosec}^2 \alpha - 1) \right)$$

$$= R^2 \cot^2 \alpha \left(\frac{\pi}{4} \cos \alpha - 1 \right) \quad \begin{array}{l} \text{(use } \beta \\ \cot^2 \alpha = \operatorname{cosec}^2 \alpha - 1) \\ \text{o.e.} \end{array}$$

(cso) A1 (4)

(e) In the given range $R^2 \cot^2 \alpha > 0$

In the interval $(0, \pi/4)$; $\frac{\pi}{4} \cos \alpha - 1$ is a decreasing function ($\because \cos \alpha$ is decreasing).

$$\text{At } \alpha = 0, \frac{\pi}{4} \cos \alpha - 1 = \frac{\pi}{4} - 1 < 0$$

$$\therefore \frac{\pi}{4} \cos \alpha - 1 < 0 \text{ in } (\pi/6, \pi/4)$$

$$\therefore \frac{dS}{d\alpha} < 0 \text{ throughout the interval.}$$

(convincing argument) M1

\therefore least value of S occurs at $\alpha = \pi/4$

A1

$$\text{Min } S = R^2 \left(\frac{\pi}{4} + 1 - \frac{\pi}{4} \cdot \sqrt{2} - \frac{\pi}{4} \cdot \frac{1}{\sqrt{2}} \right)$$

$$= R^2 \left(1 + \frac{\pi}{4} \left(-1 + \sqrt{2} + \frac{1}{\sqrt{2}} \right) \right) \quad \text{o.e.}$$

A1

(3)