United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 2 : Thursday, 26 January 2017

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1, 2, 3,* 4 *in order, and the cover sheet at the front.*
- To accommodate candidates sitting in other time zones, please do not discuss any aspect of the paper on the internet until 8am GMT on Friday 27 January.

In early March, twenty students eligible to represent the UK at the International Mathematical Olympiad will be invited to attend the training session to be held at Trinity College, Cambridge (30 March-3 April 2017). At the training session, students sit a pair of IMO-style papers and eight students will be selected for further training and selection examinations. The UK Team of six for this year's IMO (to be held in Rio de Janeiro, Brazil 12–23 July 2017) will then be chosen.

Do not turn over until **told to do so.**

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2016/17 British Mathematical Olympiad
Round 2

1. This problem concerns triangles which have vertices with integer coordinates in the usual x, y-coordinate plane. For how many positive integers $n < 2017$ is it possible to draw a right-angled isosceles triangle such that exactly n points on its perimeter, including all three of its vertices, have integer coordinates?

2. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to the real number x. Consider the sequence $a_1, a_2, \ldots$ defined by
$$a_n = \frac{1}{n}\left(\left\lfloor \frac{n}{1} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \cdots + \left\lfloor \frac{n}{n} \right\rfloor\right)$$
for integers $n \geq 1$. Prove that $a_{n+1} > a_n$ for infinitely many n, and determine whether $a_{n+1} < a_n$ for infinitely many n.

   [Here are some examples of the use of $\lfloor x \rfloor$*:* $\lfloor \pi \rfloor = 3$, $\lfloor 1729 \rfloor = 1729$ *and* $\lfloor \frac{2017}{1000} \rfloor = 2$.*]*

3. Consider a cyclic quadrilateral $ABCD$. The diagonals AC and BD meet at P, and the rays AD and BC meet at Q. The internal angle bisector of angle $\angle BQA$ meets AC at R and the internal angle bisector of angle $\angle APD$ meets AD at S. Prove that RS is parallel to CD.

4. Bobby's booby-trapped safe requires a 3-digit code to unlock it. Alex has a probe which can test combinations without typing them on the safe. The probe responds *Fail* if no individual digit is correct. Otherwise it responds *Close*, including when all digits are correct. For example, if the correct code is 014, then the responses to 099 and 014 are both Close, but the response to 140 is Fail. If Alex is following an optimal strategy, what is the smallest number of attempts needed to guarantee that he knows the correct code, whatever it is?