United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 1 : Friday, 1 December 2017

Time allowed $3\frac{1}{2}$ hours.

Instructions

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.*
- *Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.*
- *The use of rulers, set squares and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by your solutions in question number order.*
- *Staple all the pages neatly together in the top* left *hand corner.*
- *To accommodate candidates sitting in other time zones, please do not discuss the paper on the internet until 8am GMT on Saturday 2 December when the solutions video will be released at* `https://bmos.ukmt.org.uk`

Do not turn over until **told to do so.**

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2017/18 British Mathematical Olympiad

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1. Helen divides 365 by each of $1, 2, 3, \ldots, 365$ in turn, writing down a list of the 365 remainders. Then Phil divides 366 by each of $1, 2, 3, \ldots, 366$ in turn, writing down a list of the 366 remainders. Whose list of remainders has the greater sum and by how much?

2. In a 100-day period, each of six friends goes swimming on exactly 75 days. There are n days on which at least five of the friends swim. What are the largest and smallest possible values of n?

3. The triangle ABC has $AB = CA$ and BC is its longest side. The point N is on the side BC and $BN = AB$. The line perpendicular to AB which passes through N meets AB at M. Prove that the line MN divides both the area and the perimeter of triangle ABC into equal parts.

4. Consider sequences $a_1, a_2, a_3, \ldots$ of positive real numbers with $a_1 = 1$ and such that
$$a_{n+1} + a_n = (a_{n+1} - a_n)^2$$
for each positive integer n. How many possible values can a_{2017} take?

5. If we take a 2×100 (or 100×2) grid of unit squares, and remove alternate squares from a long side, the remaining 150 squares form a 100-*comb*. Henry takes a 200×200 grid of unit squares, and chooses k of these squares and colours them so that James is unable to choose 150 uncoloured squares which form a 100-comb. What is the smallest possible value of k?

6. Matthew has a deck of 300 cards numbered 1 to 300. He takes cards out of the deck one at a time, and places the selected cards in a row, with each new card added at the right end of the row. Matthew must arrange that, at all times, the mean of the numbers on the cards in the row is an integer. If, at some point, there is no card remaining in the deck which allows Matthew to continue, then he stops.

   When Matthew has stopped, what is the smallest possible number of cards that he could have placed in the row? Give an example of such a row.