

Discussion Of Various Problems In The Trinity College Admissions Quiz

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1 Foreword

In this short article we will discuss the questions which we know were part of the 2003 Trinity College Interview *tests* for Mathematics. These tests consisted of 10 short problems, of which the candidate is expected to attempt, and complete or nearly complete 2 or 3 of the problems. It is the candidate's answers to these questions that provides the basis for discussion in interview. We found that the questions from this year are considerably harder than those found in the specimen paper on the Trinity College Web-site, and think that solutions and discussions of them would be beneficial to people hoping to study maths at a Cambridge college which adopts the same system as Trinity, Clare being one example. The mathematics tends not to be exceptionally difficult, especially the pure side should be easy for a further mathematician. However, a mechanics question tends to be very hard for someone who hasn't done either A-Level Physics or M2, so it's perhaps advisable to stick to the pure questions and the statistics questions, which tend to focus on probability and combinatorics. The questions are likely to be difficult and to stretch your mathematics and challenge you for the whole hour. Myself and the other writers will discuss their interview experiences later on in this article. However, we hope that these problems and their solutions are the most beneficial thing contained in this article, and we sincerely encourage readers to try and solve all the problems before looking at the answers.

2 The Problems

Here is a selection of questions from this year's interview tests, or questions that we feel are of the same standard¹, the instructions to candidates have also been written just below:

There are ten questions below which are on various areas of mathematics. They are of varying levels of difficulty: some should be easy and others may be hard. You are not expected to answer all of them, or necessarily to give complete answers to questions. You should just attempt those that appeal to you, and they will be used as a basis for discussion in the interview that follows. You should bring the question paper, as well as what you have written, with you to the interview afterwards.

1. Which is greater as n gets large, $f(n) = 2^{2^{2^n}}$ or $g(n) = 100^{100^n}$?
2. Let $I_n = \int_0^1 x^n / \sqrt{x^3+1} dx$. Show that $(2n-1)I_n + 2(n-2)I_{n-3} = 2\sqrt{2}$. Then compute I_8 .
3. Show that if four distinct points of the curve $y = 2x^4 + 7x^3 + 3x - 5$ are collinear then their average x -coordinate is some constant k . Find k .
4. By sketching appropriate graphs, find all solutions to the equation $x - 1 = (e - 1) \log x$. Hence sketch the graph of $f(x) = e^x - x^e$. (Here, $\log x$ denotes the logarithm to base e - you may be more used to the notation $\ln x$.)
5. Six identical-looking coins are in a box, of which five are unbiased, while the sixth comes up heads with probability $3/4$ and tails with probability $1/4$. Three coins are chosen from the box at random and removed. One of those three is chosen at random and tossed three times, coming up heads every time.
 - (a) What is the probability that the final coin selected was the biased coin, given that it came up heads each time?
 - (b) What is the probability that the biased coin is amongst the three coins removed from the box?
6. Show that $\cos(n\theta) = f_n(\cos\theta)$ for polynomials $f_n(x)$ satisfying $f_{n+1}(x) = 2xf_n(x) - f_{n-1}(x)$.

¹Those that were not taken from the interview test are marked with an *asterisk*

Find all the roots of $f_2(x) + f_3(x) = 0$ and write them in the form $\cos(\phi)$ for suitable ϕ .

7. Let a and n be integers greater than 1. Suppose that $a^n - 1$ is prime. Show that $a = 2$ and n is prime. What can you say about primes of the form $2^n + 1$?

8. Consider a regular pentagon with vertices (in clockwise order) A, B, C, D, E . Let A' be the point of intersection of BD and CE , let B' be the point of intersection of CE and DA and so on. If the triangle $AC'D'$ has area 1, what is the area of pentagon $A'B'C'D'E'$?

9. A hand of thirteen playing cards is dealt from a standard pack of fifty-two. Write down expressions (in terms of binomial coefficients) for the probabilities of the following happening:

- (a) the hand contains exactly one king;
- (b) the hand contains at least two queens;
- (c) the hand contains the same number of kings as queens.

[A binomial coefficient is a number of the form $\binom{n}{k} = {}^nC_k = n!/k!(n-k)!$.]

10. Consider a mass m at position $x(t)$ on a rough horizontal table attached to the origin by a spring with constant k (restoring force $-kx$) and with a dry friction force f

$$f = \begin{cases} F & \text{if } \dot{x} < 0 \\ -F \leq f \leq F & \text{if } \dot{x} = 0 \\ -F & \text{if } \dot{x} > 0. \end{cases}$$

What is the range of x where the mass can rest? Show that if the mass moves, the maximum excursion decreases by $2F/k$ per half cycle. Discuss the motion.

11.* Assume F_{n+1}/F_n tends to a particular limit, call this x , where F_n is the sequence defined by $F_0=1, F_1=1, F_{n+2} = F_{n+1} + F_n$. Show that $1 \leq x \leq 2$, and, furthermore, determine the value of x .

3 Discussion of the Problems.

11. We may first of all determine bounds on the limit by rewriting F_{n+1}/F_n as $(F_n + F_{n-1})/F_n = 1 + F_{n-1}/F_n$. Now $F_n \geq F_{n-1}$ and thus $F_{n-1}/F_n \leq 1$, assuming a limit exists it is less than 2 but greater than 1.

Now, we may consider that if $F_{n+1}/F_n \rightarrow x$ then by our earlier rearranging we have $x = 1 + F_{n-1}/F_n$. But if $F_n/F_{n-1} \rightarrow x$ then $F_{n-1}/F_n \rightarrow 1/x$. Thus we see $x = 1 + 1/x$ and solving we see that $x = (1+\sqrt{5})/2$.

4 Personal Experiences Of The Interview

Having done a reasonable amount of preparation for the interview, most of which was in fact done indirectly through doing STEP papers and British Mathematical Olympiad questions, I felt reasonably confident before my interview. I found it useful to have read several general books from the Departmental Reading List as well, just to make sure I'd feel comfortable talking and having a discussion on more general mathematics and mathematicians. But you really only read books that you want to read, there is no point reading something that you don't want to read and are not going to remember. Anyway., I was rather shocked by the standard of difficulty of the test. This did shock me slightly, but I thought it was important to try and solve completely a few of the problems, despite them requiring a great deal of thought and working. Although the questions were long, they only required quite easy mathematics, and a good bit of insight was all that was needed to see through most of the questions. I came out of the interview glad to have done four questions, and started a fifth. In interview, one of my answers was immediately revealed to be incorrect, whereas two others were pushed aside, since my interviewers decided they were both correct. The trouble came on the fourth of my four answers, where I had asserted something that had to be proved. Here the interviewers did the best they could to guide me through the proof, seeing if I could see each step and where it was leading, testing my insight, and helping me out whenever I seemed to be stuck by providing hints to guide me towards the answer. Do not be afraid to tell them if you get stuck or do not know what to do. To admit that you do not know something is *far* better than to be stuck there, saying nothing. If you are working through a problem, try to let the professors know what you are thinking, and do not be afraid to challenge them on their views if you do not think they

are correct. Aside from going through the questions from the pre-interview test, you will probably talk about your school mathematics department and various extra-curricular mathematics activities you have been involved in and mentioned on your personal statement, but this will only be for a minute or two. It is a good idea to have some questions to ask them at the end, but if you have nothing relevant to say, say nothing, rather than posing a silly and pointless question.

Non-academically, the rest of my stay was quite enjoyable. The accomodation provided was very nice, the food was lovely (and free!) and the staff were very friendly and approachable on the whole. I also had a chance to talk to some applicants and current students in the JCR that evening, which was better than spending the whole evening alone in my room wondering how badly I thought that the interview had gone. All the people I talked to were friendly, and all seemed be as nervous as I was!

The whole application process, from start to finish, is quite a nerve-wracking procedure. The only advice I think is really worthwhile heeding is to be yourself, and go into it with a positive manner, rather than, as some people do, thinking that Cambridge is too good for you before you go into interview.

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