Stage 3 Higher Revision Sheet

This document attempts to sum up the contents of the Higher Tier Stage 3 Module.

There are two exams, each seventy five minutes long. One allows use of a calculator and the other doesn't. Together they represent 50% of the GCSE. *It is hugely important to remember that the Stage 3 exams can include all the material from Stage 1 and 2. You must revise them also!*

Before you go into the exam make sure you are fully equipped with two pens, two pencils, a calculator, a ruler, a protractor and a pair of compasses. Also be sure not to panic; it is not uncommon to get stuck on a question (I've been there!). Just continue with what you can do and return at the end to the question(s) you have found hard. If you have time check all your work, especially the first question you attempted... always an area prone to error.

I am always available on `jonathan.m.stone@gmail.com` to answer any questions you may have. Please do not hesitate.

JMS

Equations

- The most general way to solve a set of simultaneous equations (two equations in two unknowns) is to obtain a statement of x or y from one of the equations and substitute this into the other one. For example solve

  $$\begin{aligned} x+2y&=1 \\ x^2+y^2&=29 \end{aligned} \quad\Rightarrow\quad x=1-2y \quad\Rightarrow\quad (1-2y)^2+y^2=29 \quad\Rightarrow\quad (5y-14)(y+2)=0.$$

  So $y=-2$ or $y=2.8$. We then find the associated x values to find $(5,-2)$ and $(-4.6,2.8)$.

- You should also be able to appreciate the geometry of the solution of simultaneous equations as points in the xy-plane. In the above example the two points represent a line crossing a circle. The equation $x^2+y^2=r^2$ represents a circle of radius r with centre $(0,0)$.[1]

Powers, Proportion and Calculator Methods

- We know from Stage 2 that $\sqrt{ab}=\sqrt{a}\times\sqrt{b}$ and $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$. In Stage 3 you are expected to be fluent with these. For example write $\sqrt{40}$ in the form $a\sqrt{10}$; we notice $\sqrt{40}=\sqrt{4\times 10}$ so this becomes $\sqrt{4}\times\sqrt{10}=2\sqrt{10}$.

- You must be able to manipulate surd expressions. For example multiply out and simplify $(3+5\sqrt{2})^2-(2-\sqrt{2})^2 \Rightarrow (9+30\sqrt{2}+50)-(4-4\sqrt{2}+2)=53+34\sqrt{2}$.

- You must be able to rationalize the denominator of an expression. This is done by *multiplying by one* in a cunningly chosen form. In general if you have $\frac{\text{(something)}}{\sqrt{k}}$ you multiply by $\frac{\sqrt{k}}{\sqrt{k}}$.

[1]For future A Level students you should know that $(x-a)^2+(y-b)^2=r^2$ represents a circle radius r with centre (a,b). Notice how both equations look like Pythagoras statements; they are!

- In harder examples you must flip one of the signs on the bottom line. So if you have $\frac{\text{(something)}}{a+b\sqrt{k}}$ you multiply by $\frac{a-b\sqrt{k}}{a-b\sqrt{k}}$. If $\frac{\text{(something)}}{a-b\sqrt{k}}$ then multiply by $\frac{a+b\sqrt{k}}{a+b\sqrt{k}}$.
- Two examples:

$$\frac{20}{\sqrt{5}} = \frac{20}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{20\sqrt{5}}{5} = \boxed{4\sqrt{5}},$$

$$\frac{2+\sqrt{3}}{4-\sqrt{3}} = \frac{2+\sqrt{3}}{4-\sqrt{3}} \times \frac{4+\sqrt{3}}{4+\sqrt{3}} = \frac{11+6\sqrt{3}}{4^2-3} = \boxed{\frac{11}{13} + \frac{6}{13}\sqrt{3}}.$$

- We use the '$\propto$' symbol for *proportion*. However in mathematics we like the '$=$' sign. We therefore immediately replace the '$\propto$' by '$= k$' where k is a constant.

  So if y is directly proportional to x we write $y \propto x \quad \Rightarrow \quad y = kx$. Also if y is inversely proportional to the square of x we write $y \propto \frac{1}{x^2} \quad \Rightarrow \quad y = \frac{k}{x^2}$.

- After replacing the '$\propto$' by the '$= k$' we then look at the question. We will *always* be given one *pair* of values. We place these values into the equation and discover k. Once this is done we have an equation that completely defines the system.
- For example: T is proportional to the square of p. When $T = 45$, $p = 3$. (i) Find a relationship between T and p. (ii) Find T when $p = 10$. (iii) Find p when $T = 20$.

  (i) We know $T \propto p^2 \quad \Rightarrow \quad T = kp^2$. We put in values and find k. $45 = 3^2 \times k$, so $k = 5$. Therefore $\boxed{T = 5p^2.}$

  (ii) $T = 5 \times 10^2 = 500$.

  (iii) $20 = 5p^2$, so $p = 2$. (I suppose it's ± 2, but with proportion quantities tend to be physical; so positive.)

- Calculator methods are easy. If in doubt put in a bracket. For example to calculate

  $\frac{\sqrt{1.2 \times 3.4}}{56 + 7.8}$, I would type $(\sqrt{}(1.2 \times 3.4)) \div (56 + 7.8)$ into my calculator.

Straight Line and Transformed Graphs

- The usual form for a straight line graph is $y = mx + c$ where m is the gradient[2] and c is the y-intercept (i.e. where the line crosses the y-axis). See STAGE 2 REVISION SHEET for slightly fuller discussion. You *must* be able to draw a line in this form *fast*!
- A line may be given in another form. For example $ax + by = c$ and $ax + by + c = 0$ are also straight lines. These can be drawn in two ways:

  1. Rearrange the equation into the form $y = mx + c$ and then either interpret m and c or draw a table of x and y. For example $3x + 5y = 20 \quad \Rightarrow \quad y = -\frac{3}{5}x + 4$.
  2. Only two points are needed to draw a line. Let $x = 0$ and discover y to locate one point and then let $y = 0$ and discover x. For example $4x - 3y = 24$; when $x = 0$, $y = -8$ so it passes through $(0, -8)$ and when $y = 0$, $x = 6$ so it passes through $(6, 0)$.

- A few examples:

LINE	m	c	y-INTERCEPT	GRADIENT DESCRIPTION
$y = 2x - 3$	2	-3	Crosses y-axis at $(0, -3)$	From $(0, -3)$ go across right 1, up 2
$y = 2 - 3x$	-3	2	Crosses y-axis at $(0, 2)$	From $(0, 2)$ go across right 1, down 3
$y = -\frac{3}{5}x + 4$	$-\frac{3}{5}$	4	Crosses y-axis at $(0, 4)$	From $(0, 4)$ go across right 5, down 3

[2]It is a small point, but the gradient of the line $y = 3x + 4$ is 3, *not* $3x$!

- Please remember that the points that make up the straight line[3] you draw are all the *solutions* of the equation of the line. So if you are told the line $y = ax + 3$ goes through the point $(4, 11)$ then we know $11 = 4a + 3$, so $a = 2$.

- If a line has gradient m then the line at right angles (perpendicular) to it has gradient $-\frac{1}{m}$. (Flip the gradient and change the sign.) For example perp to $y = -2x + 4$ has gradient $\frac{1}{2}$.

- (A *function* is something into which we feed a number (or collection of numbers[4]) and it gives us back a single number. We write $f(x)$ to mean a function of x only. We feed in x and we get given $f(x)$ back. If $f(x) = x^2 - x + 1$ then when we feed in 10 we get back $f(10) = 10^2 - 10 + 1 = 91$. All the things you can draw (lines, quadratics, trig graphs, etc) are functions; in the most general way, when we draw *any* graph we draw $y = f(x)$.)

- Graphs of $y = f(x)$ can be transformed[5] in the following ways;

FUNCTION	GRAPH SHAPE
$f(x)$	Normal Graph
$2f(x)$	Graph stretched by a factor of 2 away from the x-axis i.e. every y value in the original graph is multiplied by 2
$f(2x)$	Graph squeezed by factor of 2 towards the y-axis i.e. every x value in the original graph is divided by 2
$3f(4x)$	Graph squeezed by factor of 4 towards the y-axis followed by stretching by a factor of 3 away from the x-axis
$f(x) + 6$	Graph translated vertically *up* 6 units
$f(x) - 6$	Graph translated vertically *down* 6 units
$f(x + 4)$	Graph translated 4 units to the *left*
$f(x - 6)$	Graph translated 6 units to the *right*
$f(x - 6) + 9$	Graph translated 6 units to the *right* and 9 units *up*. This is a translation and can be expressed as $\binom{6}{9}$ where $\binom{\text{change in } x}{\text{change in } y}$
$-f(x)$	Graph reflected in the x-axis
$f(-x)$	Graph reflected in the y-axis

- With quadratic graphs completing the square is important because it tells you the turning point (vertex) of the graph. I urge you to read the separate quadratics handout for how to do this. Here is a (relatively) hard example:

$$\begin{aligned} y &= 3 + 8x - x^2 \\ &= -x^2 + 8x + 3 \\ &= -1[x^2 - 8x] + 3 \\ &= -1[(x-4)^2 - 16] + 3 \\ &= -1(x-4)^2 + 16 + 3 \\ &= -(x-4)^2 + 19. \end{aligned}$$

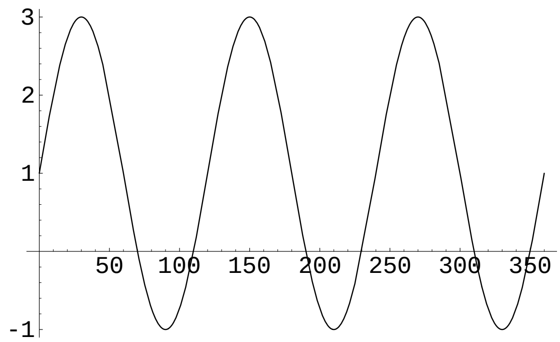
From this we can see that the turning point (in this case a maximum because it is a $\bigcap$-shaped curve) is $(4, 19)$. If you are asked to describe the transformation that maps $y = x^2$ onto this curve then it would be a reflection in the x-axis (to turn $\bigcup$ into $\bigcap$) and then a translation of $\binom{4}{19}$ (to get the vertex to the right point).

[3]. . . or quadratic or cubic or circle or trigonometric or. . .

[4]Above GCSE we can have functions of more than one variable. For example $f(x, y, z) = x + y^2 + 3z$. Therefore $f(1, 2, 3) = 1 + 2^2 + 9 = 14$.

[5]Eleanor you *can* do this!

- With trig functions it is usually pretty obvious what to do; for example to draw $y = 2\sin(3x) + 1$ then we would take a graph of $y = \sin x$ and then compress by factor of 3 to the y-axis and then stretch away from the x-axis by factor 2 and finally move it up one to get;



Could you have done this question in reverse? It does come up.

- One final (hard) example. Find the equation of the curve when $y = 3x^2 - 12x + 1$ is reflected in the line $x = -1$. We first need to find the vertex of the quadratic; so we complete the square to find $y = 3(x-2)^2 - 11$. Therefore the turning point (a minimum) is $(2, -11)$. When this point is reflected in $x = -1$ it goes to $(-4, -11)$. The new equation is therefore $y = 3(x+4)^2 - 11 = 3x^2 + 24x + 37$.

Trigonometric Functions

- Recap from Stage 1. Standard 'SohCahToa' trigonometry only works for right angled triangles. It is applicable where you are 'interested in' two lengths and one angle; one of which you don't know and would like to find. Make sure this is second nature!

- The convention for the cosine and sine rules is that angle A is opposite side a. Angle B is opposite side b. Angle C is opposite side c.

- The *cosine rule* is used when you are 'interested in' all three lengths and one angle; one of which you don't know and would like to find. It states

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

The length a is the one opposite the angle A (the only angle of interest). This form is used when the angle is known. For example if $b = 6$, $c = 9$ and $A = 40°$, find a.

$$\begin{aligned} a^2 &= 6^2 + 9^2 - 2 \times 6 \times 9 \times \cos 40 \\ a^2 &= 34.26720014\ldots \\ a &= 5.854 \text{ (to 3dp)}. \end{aligned}$$

- It can be rearranged to give

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

This form is used when the angle is unknown. For example if $a = 4$, $b = 10$ and $c = 11$, find A.

$$\begin{aligned} \cos A &= \frac{10^2 + 11^2 - 4^2}{2 \times 10 \times 11} \\ A &= \cos^{-1}\left(\frac{205}{220}\right) \\ A &= 21.28° \text{ (to 2dp)}. \end{aligned}$$

- At GCSE the cosine rule is never ambiguous[6]. However, as we will see, the sine rule can have an ambiguous solution.

- The *sine rule* is used when you are 'interested in' two lengths and two angles, with the two sides opposite the two angles; one of which you don't know and would like to find. It states (ignore the third one in the bracket)

$$\frac{\sin A}{a} = \frac{\sin B}{b} \left(= \frac{\sin C}{c} \right).$$

  This can be 'flipped' to give $\frac{a}{\sin A} = \frac{b}{\sin B} \left(= \frac{c}{\sin C}\right)$ if this is more useful[7]. For example if $a = 6$, $A = 80$ and $B = 25°$, find b.

$$\begin{aligned} \frac{b}{\sin 25} &= \frac{6}{\sin 80} \\ b &= \frac{6 \sin 25}{\sin 80} \\ b &= 2.575 \text{ (to 3dp).} \end{aligned}$$

- If a length is unknown the solution is not ambiguous. However, if an angle is unknown, then there *can* be two solutions[8]. This is because the sine rule will yield a final equation such as $\sin A = 0.9$. The primary solution to this is $A = \sin^{-1}(0.9) = 64.16°$, but (if you consider the graph or $y = \sin x$ and $y = 0.9$) $180 - 64.16° = 115.84°$ is also a potential solution. You must look at the triangle and see if this second solution is possible. If this second solution pushes the angle sum of the triangle above 180° then it is not valid. Otherwise it is. For example if $a = 3$, $A = 20$ and $b = 5$, find B.

$$\begin{aligned} \frac{\sin B}{5} &= \frac{\sin 20}{3} \\ \sin B &= \frac{5 \sin 20}{3}. \end{aligned}$$

  So 34.75° is the primary solution from your calculator, but $180 - 34.75 = 145.25°$ is also a valid solution, because it doesn't push the angle sum above 180°.

- You must be able to sketch the graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$. The first two repeat every 360°. The tan curve repeats every 180°.

- To solve trigonometric equation you *must* draw a graph after manipulating the equation to make it simpler. For example solve $5 \cos 2x + 1 = 3$ for $0 < x < 180$. We manipulate to make it simpler:

$$\begin{aligned} 5 \cos 2x + 1 &= 3 \\ 5 \cos 2x &= 2 \\ \cos 2x &= 0.4. \end{aligned}$$

  You would then dray $y = \cos 2x$ (cos wave compressed by factor two towards y-axis) and $y = 0.4$ (flat line).

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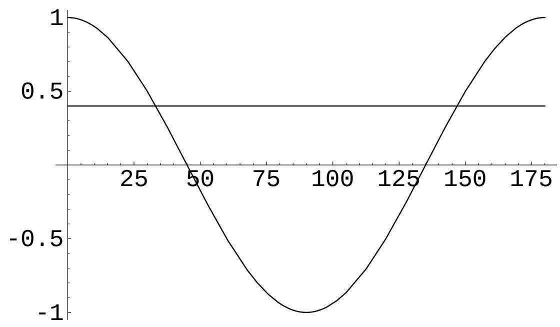
[6] The A* candidate might like to know that if it is the length b or c that is unknown then a quadratic is formed which (as I'm sure you know) can have two distinct solutions. If it is merely a or A that is unknown, then the solution is unique.

[7] Because $\frac{2}{3} = \frac{4}{6} \quad \Leftrightarrow \quad \frac{3}{2} = \frac{6}{4}$.

[8] I'm not 100% sure if this ambiguity is needed at GCSE. There was a past paper question which asked for the perimeter of a triangle; the calculation involved the sine rule and the solution was ambiguous. But I'm pretty sure they didn't require students to find both solutions; just the primary solution. *If you don't really get this point I wouldn't lose too much sleep!*



To get the primary solution you would use your calculator:

$$\cos 2x = 0.4 \quad \Rightarrow \quad 2x = \cos^{-1}(0.4) = 66.42\ldots \quad \Rightarrow \quad x = 33.21 \text{ (2dp)}.$$

But looking at the graph we can see there are two solutions in the range. Thinking about what the curve does we can soon see the second solution is $x = 180 - 33.21 = 146.79$.

- There is no such thing as 3D trig. There are no new theorems or rules. All you do with 3D shapes is visualise the shape in your mind and find 2D triangles that lie within planes in the shape itself. You can then use trigonometry and Pythagoras to your hearts content. Just be careful, it is easy to be fooled!

Circle Theorems

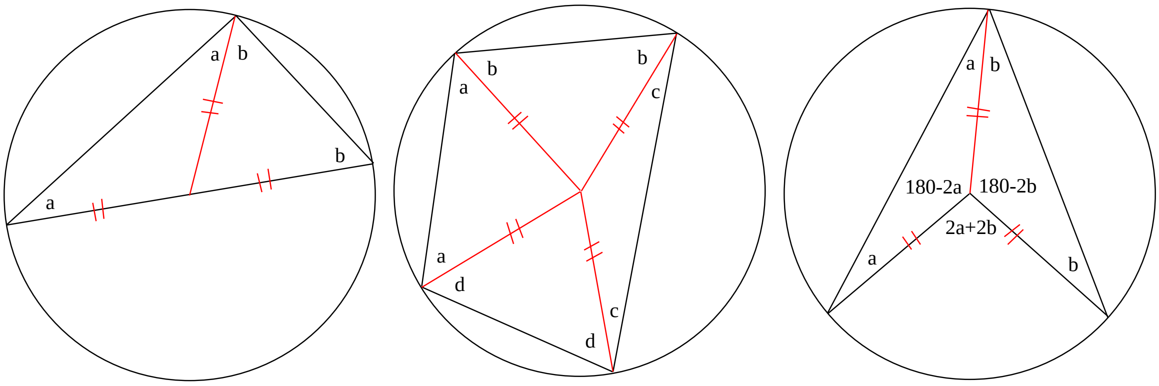
- Circle theorems are very geometric and therefore hard for me to do in this revision sheet. I will refer to the Stage 3 Book a lot. You must be able to apply *and name* the circle theorems on page 48, 49 and 51. They are:

  1. Angle in semicircle is a right angle.
  2. Angle subtended at the centre is twice the angle at the circumference.
  3. Angles in same segment are equal.
  4. Opposite angles in cyclic quadrilateral sum to 180°.
  5. The angle between a tangent and its chord is equal to the angle in the alternate segment.

  Pictures representing all these are given on Pages 55/56. Note that [3] is implied by [2]. Also [1] is a subset of [2]. Also [4] implies [3]. Think about it.

- To *prove* these results draw a sketch of the situation with the circle. Then you must put a dot at the centre of the circle and draw in as many radii that make sense. They are all the same length so show this. You will probably create a few isosceles triangles. Then denote the 'base' angles of these isosceles triangles as a, b, c, d etc.

  Three examples are given by means of the following pictures:



(a) (b) (c)

The black ink is what would be given in the paper; the red ink is what I would add to the diagram in the exam. I have not proved them all here but a few to give a flavour of how to do these proofs.

(a) Radii drawn. All same length. Two isosceles triangles created. Base angles a and b. The large triangle has angles a, $a+b$, and b. These must sum to 180, so $(a)+(a+b)+(b)=180$. Therefore $a+b=90$. This proves the angle at edge is right angle.

(b) Radii drawn. All same length. Four isosceles triangles created. Base angles a, b, c and d. The cyclic quadrilateral has angles $a+b$, $b+c$, $c+d$, and $a+d$. These must sum to 360, so $(a+b)+(b+c)+(c+d)+(a+d)=360$. Therefore $a+b+c+d=180$. This proves that opposite angles in the quadrilateral sum to 180. Look at opposite angles in the diagram.

(c) Radii drawn. All same length. Two isosceles triangles created. Base angles a and b. The missing angles in those two triangles are $180-2a$ and $180-2b$ respectively. The three angles at the centre of the circle must sum to 360 (angles at point) so the missing angle must be $360-(180-2a)-(180-2b)=2a+2b$. Angle at edge is $a+b$. Angle at centre is $2(a+b)$. This proves theorem.

Advanced Measure and Mensuration

- Obviously to convert from m to cm you multiply by 100. To convert the other way you divide by 100. From inch to cm a similar relationship exists, but with 2.54 as the conversion factor. However, with areas and volumes it is not so simple. Do *not* become one of the candidates who says that 1.3m^3 is 130cm^3.

- To do the above conversion you must draw a cube (because of the power of 3 on the unit) which compares the two different lengths (P57, second diagram). It must be true that a $1\text{m}\times1\text{m}\times1\text{m} = 1\text{m}^3$ is the same as a $100\text{cm}\times100\text{cm}\times100\text{cm} = 1{,}000{,}000\text{cm}^3$. Therefore the conversion factor between the two units must be 1,000,000. So 1.3m^3 is $1{,}300{,}000\text{cm}^3$.

- For areas you draw a square rather than a cube. For example to convert 30cm^2 to inches^2 we draw a square $1\text{in}\times1\text{in}=1\text{in}^2$. This is the same as $2.54\text{cm}\times2.54\text{cm} = 6.4516\text{cm}^2$. Clearly to convert to inches^2 we must divide by this number so we get 4.65in^2 (2dp).

- There will almost certainly be a question asking whether an expression represents and length, area or volume. If a, b, c, and d represent lengths and 7 and π are dimensionless then what are the following? Firstly we ignore the 7 and π (you can scribble them out if you like).

$3(a+b+c+d)$	LENGTH. It is just length plus length plus length. Think about it, 4m plus 5m plus 12m is 21m. The 3 just muliplies it to 63m and makes no difference to the unit.
$\dfrac{4(ab+bc+ac)}{\pi d}$	LENGTH. Overall top line is area for same reason as above. Bottom line Length.
$ab+c$	MEANINGLESS. You *cannot* add (or subtract) quantities of different dimensions. Watch out for this!
$\dfrac{7abc+\pi bcd}{a}$	AREA. Volume divided by length.
$7\pi a(bc+cd)$	VOLUME. Length times area.
$\dfrac{\pi a(bc+bd+ac)}{7bcd}$	DIMENSIONLESS. Volume divided by volume.

- It is possible they could make a question like this. If a and b are lengths and A and B are areas and Δ is a volume, what is the dimension of

$$\frac{(3A+4B)(a+7b)}{\Delta}?$$

  Just think about it the same as before. The top line is area times length which is volume. The bottom line is also volume. Therefore the overall expression is DIMENSIONLESS.

- As with 3D trigonometry, there is not really any such thing as 3D Pythagoras. You merely need to visualise the 2D right angled triangles within a 3D shape. The only result that is vaguely useful is for a cuboid l by w by h: The longest diagonal (d) within the shape is $d=\sqrt{l^2+w^2+h^2}$.

- You must know the difference between a sector and a segment of a circle. From STAGE 2 we know

$$\text{Area sector} = \frac{\theta}{360}\times\pi r^2 \qquad \text{and} \qquad \text{Area triangle} = \frac{1}{2}ab\sin\theta.$$

  By combining these results we find

$$\text{Area segment} = \text{Area sector} - \text{Area triangle} = \frac{\theta\pi r^2}{360} - \frac{1}{2}r^2\sin\theta.$$

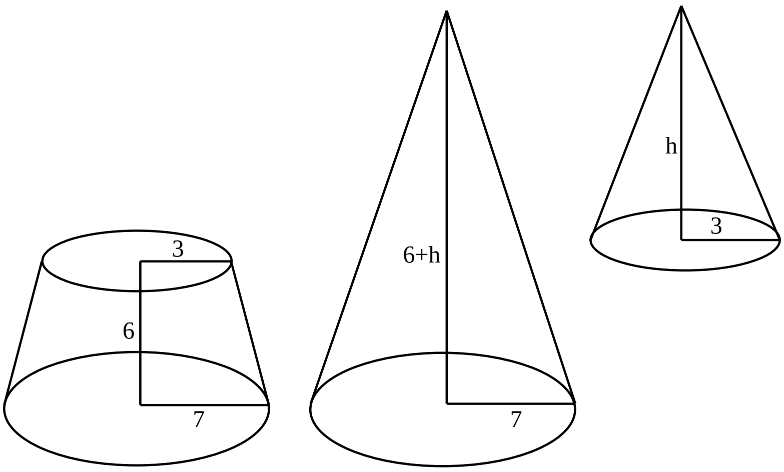
- Two shapes (2D or 3D) are said to be *mathematically similar* if they are the same shape; i.e. every ratio of two lengths is the same between the two shapes. Angles are unchanged between similar shapes.

  We call k the length scale factor. If the length on a smaller shape is 3cm and the same length on a larger similar shape is 7cm then $k=\frac{7}{3}$. So if another length on the larger shape is 20cm, then the equivalent shorter length would be $20\div\frac{7}{3}=8\frac{4}{7}$cm.

- The conversion factor for areas is k^2 and the conversion factor for volumes is k^3. For example two bottles of shampoo are similar. The area of the label on one is 50cm^2 and on another the label is 72cm^2. If the volume of the larger bottle is 864cm^3. What is the volume of the smaller bottle?

  We know $k^2=72/50=1.44$; therefore $k=1.2$. We know volumes go as $k^3=1.728$. Therefore the volume of the smaller bottle is $864\div1.728=500$cm^3.

- Compound solids usually come in the form of frustums; cones with the tip cut off. You must draw a sketch of the frustum with the cone put back on, and the little cone that has been cut off. For example find the volume of the following frustum.



We know the two cones must be similar. Therefore the right angled triangles within the cones must be similar. Therefore
$$\frac{6+h}{7} = \frac{h}{3}.$$
This solves to $h = 4.5$. Therefore the volume of the frustum is
$$V_{\text{frustum}} = V_{\text{big cone}} - V_{\text{small cone}} = \frac{1}{3}\pi r_b^2 h_b - \frac{1}{3}\pi r_s^2 h_s = \frac{1}{3}\pi(7^2 \times 10.5 - 3^2 \times 4.5) = 158\pi.$$

Handling Data